SOME TOPICS IN THE ARITHMETIC OF POLYNOMIALS

L. CARLITZ

1. **Introduction.** Let $GF(p^n)$ denote a fixed Galois (finite) field, and x an indeterminate. The arithmetic of polynomials in x with coefficients in $GF(p^n)$ is in many ways similar to ordinary arithmetic, and was discussed in some detail by Dedekind.[1] As a matter of fact it appears that in many instances the arithmetic of polynomials is the simpler. Thus, for example, in the case of the analogues of the familiar arithmetic functions, in place of asymptotic formulas there are exact formulas for the polynomial domain. This is perhaps due to the possibility of grouping polynomials according to degree. Again it is familiar that in the problem of representing a rational integer as a sum of an even number of squares there is a considerable difference between the case $2t \leqq 8$ and $2t > 8$; in the former case the number of representations can be expressed in terms of divisor functions, while in the latter case this is in general impossible. For the polynomial case however the number of representations by an even number of squares can always be expressed in terms of divisor functions. Similar remarks apply to the case of an odd number of squares.

In the present paper we rather arbitrarily select three or four topics in the arithmetic of polynomials in a Galois field. In §2 we consider the simplest arithmetic functions. In §3 we discuss the problem of representing a given polynomial as a sum of squares. In §4 we define various special polynomials and functions that are rather intimately connected with the arithmetic of polynomials in $GF(p^n)$; application to power sums are given in §5. Finally in §6 we define analogues of the ordinary Bernoulli numbers; the principal result here is the Staudt-Clausen theorem. We remark that for the most part the extension of theorems from the coefficient field[2] $GF(p)$ to $GF(p^n)$ is quite trivial; however, in at least the last topic mentioned there appears to be some difference between the special and the general case.

It is evident from the above that we are ignoring such questions as construction and distribution of irreducible polynomials, the existence of irreducibles in an arithmetic progression, theorems of reciproc-

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[1] Journal für die reine und angewandte Mathematik, vol. 54 (1857), pp. 1–26.

[2] The field $GF(p)$ may be defined as the set of residues (mod p).

ity and higher congruences generally, to mention a few obvious omissions. We give a few references below.[3]

2. **Arithmetic functions.**[4] It will be convenient to denote polynomials in x by capitals $A, B, \cdots, Z$; elements of $GF(p^n)$ and ordinary integers will be denoted by small letters. We write deg M for the degree of M and put $|M| = p^{nm}$, $m = \deg M$; if the coefficient of the highest power of x in M is 1 we call M *primary*. The letter P will be reserved for *irreducible* polynomials. Evidently the number of primary polynomials of fixed degree m is p^{nm}. Hence the ζ-function for our domain becomes

$$\zeta(s) = \sum_M \frac{1}{|M|^s} = \sum_{m=0}^{\infty} \frac{p^{nm}}{p^{nms}} = \frac{1}{1 - p^{n(1-s)}} \tag{2.1}$$

for $\Re(s) > 1$; the first summation in (2.1) is taken over all primary polynomials. Now since the unique factorization theorem applies here we have

$$\sum_M \frac{1}{|M|^s} = \prod_P \left\{1 + \frac{1}{P^s} + \frac{1}{P^{2s}} + \cdots\right\}$$

$$= \prod_P \left\{1 - \frac{1}{|P|^{-s}}\right\}^{-1} = \prod_{k=1}^{\infty} \left\{1 - \frac{1}{p^{ks}}\right\}^{-f(k)},$$

where $f(k)$ denotes the number of (primary) irreducible polynomials of degree k. Comparison with (2.1) leads to the identity

$$1 - p^{n(1-s)} = \prod_{k=1}^{\infty} (1 - p^{-ks})^{f(k)}, \tag{2.2}$$

by means of which it is easy to derive the familiar formula for $f(k)$. We remark that (2.2) can be considerably extended.

Now define the Möbius function $\mu(M)$ by means of

$$\mu(1) = 1, \qquad \mu(M) = 0 \quad \text{for} \quad P^2 \mid M,$$

$$\mu(M) = (-1)^r \quad \text{for} \quad M = P_1 \cdots P_r.$$

Then

$$\sum_M \frac{\mu(M)}{|M|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right) = \frac{1}{\zeta(s)} = 1 - p^{n(1-s)},$$

[3] See for example L. E. Dickson, *Linear Groups*, 1901, pp. 3–54; H. Kornblum, Mathematische Zeitschrift, vol. 5 (1919), p. 107; E. Artin, Mathematische Zeitschrift, vol. 19 (1924), pp. 153–246; O. Ore, Transactions of this Society, vol. 35 (1933), pp. 559–584; L. Carlitz, American Journal of Mathematics, vol. 59 (1937), pp. 618–628.

[4] Compare American Journal of Mathematics, vol. 54 (1932), pp. 39–50.

and therefore

$$\sum_{\deg M=m} \mu(M) = \begin{cases} -p^n & \text{for } m = 1, \\ 0 & \text{for } m > 1. \end{cases}$$

Similarly the number of quadrat-frei (simple) polynomials of degree m is $p^{nm} - p^{n(m-1)}$.

In exactly the same way if $\delta(M)$ denotes the number of (primary) divisors of M and $\phi(M)$ denotes the Euler function (number of residues in a reduced residue system (mod M)) then we have the identities

$$\sum_M \frac{\delta(M)}{|M|^s} = \zeta^2(s) = (1 - p^{n(1-s)})^{-2},$$

$$\sum_M \frac{\phi(M)}{|M|^s} = \frac{\zeta(s-1)}{\zeta(s)} = \frac{1 - p^{n(1-s)}}{1 - p^{n(2-s)}},$$

by means of which it is easy to evaluate

$$\sum_{\deg M=m} \delta(M), \qquad \sum_{\deg M=m} \phi(M).$$

We remark that $\delta(M)$ and $\phi(M)$ may be generalized in the following way

$$\delta_k(M) = \sum_{A|M,\ \deg A=k} 1, \tag{2.3}$$

that is, the number of divisors of given degree k. Similarly $\phi_k(M)$ denotes the number of primary polynomials of degree k, each prime to M. These functions are useful in certain problems (see, for example, (3.8) below).

3. **Sums of squares.**[5] Assume p odd. Let t be an integer greater than 0; $\alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_t$ elements of $GF(p^n)$ such that

$$\gamma_i = \alpha_i + \beta_i \neq 0, \qquad i = 1, \cdots, t. \tag{3.1}$$

Then if $\gamma = \gamma_1 + \cdots + \gamma_t \neq 0$, and M is primary of even degree $2k$, we seek the number of solutions of

$$\gamma M = \alpha_1 X_1^2 + \beta_1 Y_1^2 + \cdots + \alpha_t X_t^2 + \beta_t Y_t^2 \tag{3.2}$$

in primary X_i, Y_i of degree k. We call the problem (A).

Next if $\gamma_1 + \cdots + \gamma_t = 0$, $\alpha \neq 0$, and M is primary of degree less then $2k$, we seek the number of solutions of

[5] Compare Transactions of this Society, vol. 35 (1933), pp. 397–410; Duke Mathematical Journal, vol. 1 (1935), pp. 298–315.

$$\alpha M = \alpha_1 X_1^2 + \beta_1 Y_1^2 + \cdots + \alpha_t X_t^2 + \beta_t X_t^2. \tag{3.3}$$

We call this problem (B).

The solution of these problems is given in terms of certain "divisor" functions now to be defined. For brevity we limit ourselves to the case of problem (A). Put

$$\rho_t(M) = \left(1 - \frac{1}{p^{nt}}\right) \sum_{A|M}^{a>k} |A|^t + \sum_{A|M}^{a=k} |A|^t, \tag{3.4}$$

where $a = \deg A$, $|A| = p^{na}$, the first summation is over all primary A dividing M and of degree greater than k, the second is over all primary A dividing M and of degree equal to k;

$$\omega_t(M) = \left(1 + \frac{1}{p^{nt}}\right) \sum_{A|M}^{a>k} (-1)^a |A|^t + \sum_{A|M}^{a=k} (-1)^a |A|^t, \tag{3.5}$$

the notation having the same meaning as in (3.4). *Then the number of solutions of* (3.2) *is* $\rho_{t-1}(M)$ *or* $\omega_{t-1}(M)$ *according as*

$$(-1)^t \alpha_1 \cdots \alpha_t \beta_1 \cdots \beta_t$$

is or is not a square in $GF(p^n)$.

The proof of this theorem is by induction. The case $t=1$ is easily proved. For the rest it suffices to prove the following three formulas

$$\begin{aligned} \sum \rho_s(A)\rho_t(B) &= \rho_{s+t+1}(M), \\ \sum \rho_s(A)\omega_t(B) &= \omega_{s+t+1}(M), \\ \sum \omega_s(A)\omega_t(B) &= \rho_{s+t+1}(M), \end{aligned} \tag{3.6}$$

where the summation is over all primary A, B of degree $2k$ such that

$$(\alpha + \beta)M = \alpha A + \beta B, \tag{3.7}$$

and α, β are elements of $GF(p^n)$, $\alpha\beta(\alpha+\beta) \neq 0$. We sketch briefly a proof of the first of (3.6); by introducing certain additional notation we may prove all three formulas simultaneously.

It is convenient to make use of the divisor function $\delta_t(M)$ defined in (2.3). For this function we have the theorem[6]

$$\sum \delta_i(A)\delta_j(B) = (1 - p^{-n}) \sum_{a=i+1}^{2k} \delta_a(M) + \delta_i(M), \tag{3.8}$$

where $k \leqq j \leqq i \leqq 2k$, and the summation on the left is over all A,

[6] Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 116–124, in particular, p. 122.

B satisfying (3.7). Then by (3.4)

$$\sum \rho_s(A)\rho_t(B) = \epsilon_s\epsilon_t \sum_{i,\,j=k+1}^{2k} p^{n(is+jt)}g(i,\,j) + p^{nk(s+t)}g(k,\,k) + \epsilon_s p^{nkt} \sum_{i=k+1}^{2k} p^{nis}g(i,\,k) + \epsilon_t p^{nks} \sum_{j=k+1}^{2k} p^{njt}g(k,\,j),$$

where $g(i, j)$ denotes the left member of (3.8) and $\epsilon_t = 1 - p^{-nt}$. Now using (3.8) the proof of the formula in question follows without much difficulty.

We have assumed p odd. For $p=2$ the right member of (3.2) or (3.3) is a perfect square and therefore the problem is of no interest. However, in this case we may consider the number of solutions of

$$\gamma M = \sum_{i=1}^{t} (\alpha_i X_i^2 + \beta_i X_i Y_i + \gamma_i Y_i^2),$$

where now $\prod \beta_i \not\equiv 0$. It may be shown that results similar to the above still hold. The final form of the result depends on whether the quadratic form $\alpha_i X^2 + \beta_i XY + \gamma_i Y^2$ is irreducible in $GF(2^n)$. As a special case of some interest we take

$$\gamma M = \beta_1 X_1 Y_1 + \cdots + \beta_t X_t Y_t, \tag{3.9}$$

where

$$\gamma = \beta_1 + \cdots + \beta_t \not\equiv 0, \qquad \beta_i \not\equiv 0.$$

Evidently the number of solutions of this problem in primary X_i, Y_i of degree k is

$$\sum \delta_k(M_1) \cdots \delta_k(M_t), \tag{3.10}$$

the summation extending over primary M_i of degree $2k$ such that $\gamma M = \beta_1 M_1 + \cdots + \beta_t M_t$. But by (3.4) and the definition of $\delta_i(M)$ it is clear that $\delta_k(M) = \rho_0(M)$, so that (3.9) becomes

$$\sum \rho_0(M_1) \cdots \rho_0(M_t) = \rho_{t-1}(M),$$

as follows from (3.6). Hence the number of solutions of (3.9) is $\rho_{t-1}(M)$. This result holds for all p.

The situation for an odd number of squares is quite different. Corresponding to (3.2) we consider

$$\epsilon M = \alpha_1 X_1^2 + \cdots + \alpha_{2t+1} X_{2t+1}^2, \tag{3.11}$$

where $\epsilon = \alpha_1 + \cdots + \alpha_{2t+1} \not\equiv 0$, while to (3.3) corresponds

$$\alpha M = \alpha_1 X_1^2 + \cdots + \alpha_{2t+1} X_{2t+1}^2, \tag{3.12}$$

where $\alpha_1 + \cdots + \alpha_{2t+1} = 0$, $\alpha \neq 0$ arbitrary.

Put $\theta = (-1)^t \epsilon \alpha_1 \cdots \alpha_{2t+1}$ or $(-1)^t \alpha \alpha_1 \cdots \alpha_{2t+1}$ according as (3.11) or (3.12) is being considered. For simplicity we limit ourselves to the case M quadrat-frei. Then the number of solutions of (3.11) or (3.12) is given by

$$p^{n(k(t-1)}\{\sigma_k + p^{nt}\sigma_{k-1} + (p^{2nt} - p^n)\sigma_{k-2} + \cdots + (p^{nkt} - p^{n(kt-2t+1)})\sigma_0\} \tag{3.13}$$

where

$$\sigma_j = \sigma_j(\theta M) = \sum_{\deg A = j} (\theta M/A),$$

summed over primary A of degree j, and $(\theta M/A)$ is the quadratic residue symbol. The proof of (3.13) depends on the previous results for (3.2) and (3.3).

We remark finally that the results of this section can be obtained by an entirely different method which applies to either an odd or even number of squares. This was suggested by Hardy's paper *On the representation of a number as the sum of any number of squares and in particular of five.*[7] However, we shall not take the space to discuss this method.

4. Special polynomials and functions.[8] Put

$$\begin{aligned} F_k &= [k][k-1]^{p^n} \cdots [1]^{p^{n(k-1)}}, & F_0 &= 1, \\ L_k &= [k][k-1] \cdots [1], & L_0 &= 1, \end{aligned} \tag{4.1}$$

where

$$[k] = x^{p^{nk}} - x.$$

Then as is well known, $[k]$ is the product of irreducible polynomials of degree a divisor of k. As for F_k and L_k, it may be shown that F_k is the product of the (primary) polynomials of degree k, while L_k is the least common multiple of these polynomials.

Next consider

$$\psi_m(t) = \prod_{\deg A < m} (t + A), \qquad \psi_0(t) = t, \tag{4.2}$$

where t is a second indeterminate and the product is over all $A = A(x)$ of degree less than m. It is not difficult to show that (4.2) implies

[7] Transactions of this Society, vol. 21 (1920), pp. 255–284.

[8] Compare Duke Mathematical Journal, vol. 1 (1935), pp. 137–168.

$$\psi_m(t) = \sum_{i=0}^{m} (-1)^{m-i} \begin{bmatrix} m \\ i \end{bmatrix} t^{p^{ni}}, \tag{4.3}$$

where the coefficients are determined by

$$\begin{bmatrix} m \\ i \end{bmatrix} = \frac{F_m}{F_i L_{m-i}^{p^{ni}}}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = \frac{F_m}{L_m}, \quad \begin{bmatrix} m \\ m \end{bmatrix} = 1,$$

and are integral, that is, polynomials in x. Of the properties of $\psi_m(t)$ that follow from (4.3) we mention

$$\psi_m(xt) = x\psi_m(t) + [m]\psi_{m-1}^{p^n}(t), \tag{4.4}$$

$$\psi_m(t) = \psi_{m-1}^{p^n}(t) - F_{m-1}^{p^n-1}\psi_{m-1}(t). \tag{4.5}$$

Also as a consequence of (4.2) we have

$$\psi_m(M) = F_m \tag{4.6}$$

for M primary of degree m. Note that

$$\psi_m(t+u) = \psi_m(t) + \psi_m(u), \ \psi_m(ct) = c\psi_m(t) \text{ for } c \text{ in } GF(p^n);$$

we therefore call $\psi_m(t)$ a "linear" polynomial.

The formula (4.4) suggests the operator Δ defined by

$$\Delta g(t) = g(xt) - xg(t);$$

Δ^k is defined recursively by

$$\Delta^{k+1}g(t) = \Delta^k g(xt) - x^{p^{nk}}\Delta^k g(t).$$

From these formulas it follows in particular that

$$\Delta^k \psi_m(t) = [m] \cdots [m-k+1]^{p^{n(k-1)}} \psi_{m-k}^{p^{nk}}(t), \qquad m \geqq k. \tag{4.7}$$

Now let $g(t) = \sum \alpha_i t^{p^{ni}}$ be any linear polynomial in t. Clearly it may be expressed in the form $\sum \beta_i \psi_i(t)$. The coefficients are readily determined by means of (4.7). Replacing t by tu we get the expansion

$$g(tu) = \sum_i (1/F_i)\psi_i(u)\Delta^i g(t). \tag{4.8}$$

Thus for $g(t) = \psi_m(t)$, (4.8) becomes

$$\psi_m(tu) = \sum_{i=0}^{m} \begin{Bmatrix} m \\ i \end{Bmatrix} \psi_i(u)\psi_{m-i}^{p^{ni}}(t), \tag{4.9}$$

while for $g(t) = t^{p^{nm}}$ we get

$$t^{p^m} = \sum_{i=0}^{m} \begin{Bmatrix} m \\ i \end{Bmatrix} \psi_i(t). \tag{4.10}$$

The coefficients in (4.9) and (4.10) are given by

$$\begin{Bmatrix} m \\ i \end{Bmatrix} = \frac{F_m}{F_i F_{m-i}^{p^i}}, \qquad \begin{Bmatrix} m \\ 0 \end{Bmatrix} = \begin{Bmatrix} m \\ m \end{Bmatrix} = 1,$$

and are integral.

The polynomial $\psi_m(t)$ suggests the construction of a function vanishing for all $A = A(x)$. We find that the function

$$\psi(t) = \sum_{i=0}^{\infty} (-1)^i t^{p^{mi}}/F_i \tag{4.11}$$

has the product expansion

$$\psi(t) = t \prod_M \left\{ 1 - \frac{t^{p^{n-1}}}{(M\xi)^{p^n-1}} \right\}, \tag{4.12}$$

the product extending over *all* primary M. Hence $\psi(A\xi) = 0$ for all A. As for ξ it may be defined by

$$\xi = \lim_{k=\infty} \frac{[1]^{p^{nk}/(p^n-1)}}{L_k}. \tag{4.13}$$

It has recently been proved[9] that ξ is transcendental relative to the field $GF(p^n, x)$, that is, the field of rational functions in x with coefficients in $GF(p^n)$.

Applying the operator Δ to $\psi(t)$ we find

$$\psi(xt) = x\psi(t) - \psi^{p^n}(t); \tag{4.14}$$

repeated use of this formula leads to the multiplication formula

$$\psi(Mt) = \sum_{i=0}^{m} (-1)^i \frac{\psi_i(M)}{F_i} \psi^{p^{ni}}(t), \tag{4.15}$$

where M is primary of degree m. This in turn suggests the introduction of the linear polynomial $\omega_M(u)$ defined by

$$\omega_M(\psi(t)) = \psi(Mt);$$

next put

$$W_M(u) = \prod_{AB=M} \{\omega_A(u)\}^{\mu(B)},$$

with $\mu(B)$ as in §2. Then $W_M(u)$ may be thought of as an analogue of the cyclotomic polynomial; in particular it is irreducible.[10]

[9] L. I. Wade, Duke Mathematical Journal, vol. 8 (1941), pp. 701–720.

[10] See Transactions of this Society, vol. 43 (1938), pp. 167–182.

Returning to (4.11) we find that the inverse of $\psi(t)$ is also of simple form, namely

$$\lambda(t) = \sum_{i=0}^{\infty} (1/L_i) t^{p^{ni}}. \tag{4.16}$$

As for the convergence of (4.11) and (4.16) if we take

$$t = c_k x^k + \cdots + c_0 + \frac{c_{-1}}{x} + \cdots, \tag{4.17}$$

where the c_j are all in $GF(p^n)$, then $\psi(t)$ converges for all t while $\lambda(t)$ converges only for $k \leqq 1$.

It follows from (4.11) and (4.12) that

$$\psi(t + M\xi) = \psi(t) + \psi(M\xi) = \psi(t), \tag{4.18}$$

for arbitrary M; in other words $\psi(t)$ has the period ξ. This property can be generalized and "linear" functions with any number of periods can be constructed. For example, if we write (4.17) in the form

$$\psi(t + M_0\xi + M_1\theta\xi + \cdots + M_{n-1}\theta^{n-1}\xi) = \psi(t),$$

where θ defines $GF(p^n)$ and the M_i have coefficients in $GF(p)$, then we may regard $\psi(t)$ as an n-ply periodic function with respect to the smaller field. For the general case we put

$$f(t) = \sum_{i=0}^{\infty} (-1)^i A_i t^{p^{ni}}/F_i; \tag{4.19}$$

in place of (4.14) we now have

$$f(xt) = xf(t) + \sum_{j=1}^{k} (-1)^j \gamma_j f^{p^{nj}}(t), \tag{4.20}$$

which is characteristic. Using (4.20) we get a recursion for A_i,

$$A_i = \sum_{j=1}^{k} \frac{F_{i-1}^{p^n}}{F_{i-j}^{p^{nj}}} \gamma_j A_{i-j}^{p^{nj}}.$$

It is easily verified that (4.19) converges for all t defined by (4.17). The multiplication formula (4.15) now becomes

$$f(Mt) = \sum_{i=0}^{mk} (-1)^i \frac{f_i(M)}{F_i} f^{p^{ni}}(t),$$

thus defining a set of linear polynomials $f_i(u)$.

5. **Power sums.**[11] Going back to (4.2) it is not difficult to show that it implies

$$\sum_{\deg M=m} \frac{1}{t+M} = (-1)^m \frac{F_m}{L_m} \frac{1}{\psi_m(t)+F_m}. \tag{5.1}$$

In this identity put $t=0$ and we get

$$\sum_{\deg M=m} (1/M) = (-1)^m/L_m, \tag{5.2}$$

so that by the remark at the beginning of §4, the sum of the reciprocals of the polynomials of degree m is, except for sign, the reciprocal of the L. C. M. If we expand (5.1) in descending powers of t it is easily seen that

$$\sum_{\deg M=m} M^{p^{nm}-1} = (-1)^m F_m/L_m \tag{5.3}$$

while

$$\sum_{\deg M=m} M^k = 0 \qquad \text{for} \quad k < p^{nm}-1. \tag{5.4}$$

(5.2) and (5.3) are special cases of[12]

$$\sum_{\deg M=m} M^{p^{nk}-1} = (-1)^m \frac{F_k}{L_m F_{k-m}^{p^{nm}}}, \qquad k \geqq m, \tag{5.5}$$

and

$$\sum_{\deg M=m} M^{1-p^{nk}} = \frac{L_{k+m-1}}{L_{k-1} L_m^{p^{nk}}}. \tag{5.6}$$

We outline a new proof of these formulas.

Consider the sum

$$\sum_{\deg M=m} M(t)/M(x). \tag{5.7}$$

Since (5.7) is a polynomial in t of degree m, it may be determined by assigning $m+1$ values to t. We take

$$t = x^{p^{ni}}, \qquad i = 0, 1, \cdots, m.$$

Using the Lagrange interpolation formula together with (5.3) and (5.4) leads to the identity

$$\sum_{M} M(t)/M(x) = ((-1)^m/L_m)(t-x)(t-x^{p^n}) \cdots (t-x^{p^{n(m-1)}}). \tag{5.8}$$

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[11] Compare Duke Mathematical Journal, vol. 5 (1939), pp. 941–947.

[12] More general formulas are proved by H. L. Lee in a Duke University thesis.

In (5.8) take $t=x^{p^{nk}}$ and we get (5.5); on the other hand interchange x and t, put $t=x^{p^{nk}}$, and the result is (5.6).

We next construct a polynomial[13] $G_m(t)$ which is useful in evaluating more general power sums. Put

$$m = a_0 + a_1 p^n + a_2 p^{2n} + \cdots, \qquad 0 \leqq a_i < p^n;$$

then define

$$G_m(t) = \psi_0^{a_0}(t)\psi_1^{a_1}(t)\psi_2^{a_2}(t)\cdots, \qquad G_0(t) = 1, \tag{5.9}$$

so that $G_m(t)$ is of degree m in t, and the coefficients are integral in x. We also define

$$g_m = F_1^{a_1}F_2^{a_2}\cdots, \qquad g_0 = 1. \tag{5.10}$$

We shall not now go into the connection between $G_m(t)$ and power sums but instead quote an application of a different sort. A polynomial $f(t)$ may be called integral-valued if $f(A)$ is integral for all integral A. Then we have the theorem:—*the polynomial*

$$\sum_{i=0}^{k} A_i G_i(t)/g_i$$

is integral-valued if and only if the coefficients A_i are integral, that is, polynomials in x.

6. **Bernoulli numbers.**[14] As analogues of the Bernoulli numbers we define a set of rational functions B_m by means of

$$\frac{t}{\psi(t)} = \sum_{m=0}^{\infty} \frac{B_m}{g_m} t^m, \tag{6.1}$$

where $\psi(t)$ is defined by (4.11) and g_m is given by (5.10). Note that B_m is defined only for m a multiple of p^n-1. Then in the first place we have the formula

$$\sum_A \frac{1}{A^m} = \frac{B_m}{g_m}\xi^m, \qquad p^n - 1 \mid m, \tag{6.2}$$

the summation extending over all primary polynomials, and ξ as in (4.13). Thus (6.2) is the analogue of a familiar formula; in particular it shows that $B_m \neq 0$. Unfortunately no formula connecting B_m with

[13] See Duke Mathematical Journal, vol. 6 (1940), pp. 486–504.

[14] Compare Duke Mathematical Journal, vol. 3 (1937), pp. 503–517; vol. 7 (1940), pp. 62–67.

finite power sums is available and therefore the usual methods for deriving arithmetic properties of the ordinary Bernoulli numbers paparently cannot be applied. Instead we make use of certain ideas due to Hurwitz.[15]

We call a series of the form

$$H(t) = \sum_{m=0}^{\infty} A_m t^m/g_m, \tag{6.3}$$

where the A_m are integral, a Hurwitz series, briefly an H-series. It follows that the sum and product of two H-series are again H-series; if $A_0=1$ then the reciprocal is an H-series. If $A_0=0$ we call (6.3) an H_1-series. For this case we have the result that H_1^k/g_k is an H-series. This result applied to

$$\psi^{p^{nk}-1}(t) = \sum_m A_m^{(k)} t^m/g_m \tag{6.4}$$

shows that $A_m^{(k)}$ is a multiple of F_k/L_k.

Now returning to the definition of B_m we have

$$\frac{t}{\psi(t)} = \frac{\lambda(\psi(t))}{\psi(t)} = \sum_{k=0}^{\infty} \frac{1}{L_k} \psi^{p^{nk}-1}$$

by (4.15) and therefore using (6.4) we get the formula

$$B_m = \sum_k \frac{1}{L_k} A_m^{(k)}. \tag{6.5}$$

From this result it follows that the denominator of B_m contains only simple factors. To improve this we require the following lemma: *Let P be irreducible of degree k. Then*

$$\psi^{p^{nk}-1}(t) \equiv \left\{ \sum_{i=0}^{\infty} (-1)^i \frac{t^{p^{nki}}}{F_{ki}} \right\}^{p^{nk}-1} \pmod{P}. \tag{6.6}$$

It is now easy to evaluate $A_m^{(k)} \pmod{P}$. Substituting in (6.5) we get the following result.

THEOREM ($p^n \neq 2$). *Let*

$$m = \sum_i a_i p^i, \qquad 0 \leqq a_i < p.$$

Then if

[15] Mathematische Annalen, vol. 51 (1899), pp. 196–226 (= Mathematische Werke II, Basel, 1933, pp. 342–373).

$$\sum_i a_i = nk(p-1), \qquad p^{nk}-1 \mid m, \tag{6.7}$$

is inconsistent, B_m is integral; while if (6.7) *is consistent, k is uniquely determined and*

$$B_m = G_m - e \sum_{\deg P=k} \frac{1}{P}, \tag{6.8}$$

where G_m is integral, the summation is over irreducible polynomials P of degree k, and

$$e = \frac{(-1)^{nk+dk}}{\prod_i (a_i!)}, \qquad d = \sum_{i,j} i a_{ink+j}.$$

The case $p^n = 2$ is covered by a supplementary theorem which we shall omit.

We remark that if the function $f(t)$ has an inverse of the form $\sum (D_i/L_i)t^{p^{ni}}$, where the D_i are integral, then for the coefficients of $t/f(t)$ there is a decomposition into partial fractions similar to (6.8). This result evidently generalizes the above theorem on B_m.

DUKE UNIVERSITY

DIFFERENTIAL EQUATIONS WITH GENERAL BOUNDARY CONDITIONS

WILLIAM M. WHYBURN

This paper concerns ordinary differential equations in the real domain. More specifically, it discusses systems of first order, nonsingular, equations of the form

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n), \qquad i = 1, 2, \cdots, n, \tag{1}$$

together with boundary conditions at one, two, or more points of the interval of definition. System (1) is quite general and can be made to include the nonsingular equations of the nth order as special cases. The paper makes no attempt to generalize the equations beyond those that occur in standard treatments. Any elements of generality or novelty introduced appear in connection with the types of boundary conditions discussed. The mathematical literature of the past century contains many results for system (1) with boundary conditions at one point—the so-called fundamental existence theorems occupying central positions—and with boundary conditions at two points of the interval where second order linear systems have been of prime importance. These important bodies of results have been suveyed before the Society through addresses by Bôcher, Bliss, Pell-Wheeler, Reid and others, and therefore are given little mention in the present paper. We discuss results that have been obtained in cases where the boundary conditions apply to n or less points, to any finite number of points, to any infinite point set of the first species,[1] and where the boundary conditions contain integrals over an interval. The literature on differential systems with such boundary conditions is not extensive although many substantial results have been obtained and potential applications exist in a number of fields.

1. **Fundamental existence theorems.** Let the real functions $f_i(x, y_1, \cdots, y_n)$ in system (1) be defined over a domain

$$D: \begin{cases} (a, b): & a \leqq x \leqq b \\ R: & A_i < y_i < B_i \end{cases} \qquad (i = 1, \cdots, n),$$

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[1] A set of the first species is one which has at most a finite number of nonvacuous derived sets. Any such set is at most denumerably infinite. See Hobson, *Theory of Functions of a Real Variable*, vol. 1, 2d edition, Cambridge, 1920, pp. 71 and 79.

of $(n+1)$-dimensional real space, where a and b are finite but any or all of the A_i's and B_i's may be infinite. By a solution of system (1) on an interval $\alpha \leqq x \leqq \beta$ we understand a set of absolutely continuous functions $y_i = g_i(x)$ $(i = 1, 2, \cdots, n)$, such that the equations $g_i' = f_i[x, g_1(x), g_2(x), \cdots, g_n(x)]$ $(i = 1, 2, \cdots, n)$, hold at all points of $\alpha \leqq x \leqq \beta$ except possibly for a set of points of measure zero. Classical fundamental existence theorems for this system are of "neighborhood" and "interval" types. The first of these uses light hypotheses but gives existence of solutions in neighborhoods of specified points only. The second or "interval" type uses strong hypotheses and yields existence throughout previously assigned intervals. A typical theorem of the first type follows:

THEOREM I. *Let the functions $f_i(x, y_1, \cdots, y_n)$ be continuous in $(y_1, \cdots, y_n)$ throughout R for each fixed x on (a, b), and measurable in x on (a, b) for fixed $(y_1, \cdots, y_n)$ in R. Let a Lebesgue integrable function $M(x)$ exist on (a, b) such that throughout D, $|f_i(x, y_1, \cdots, y_n)| < M(x)$ $(i = 1, 2, \cdots, n)$. If $P(c, k_1, k_2, \cdots, k_n)$ is any point of D, there exists a neighborhood of $x = c$ on which equation* (1) *has at least one solution such that $y_i(c) = k_i$ $(i = 1, 2, \cdots, n)$.*

Theorem I is due to Carathéodory[2] and contains the Peano type theorems where the functions $f_i(x, y_1, \cdots, y_n)$ are assumed continuous throughout D, as special cases. For the latter theorems, the uniform boundedness condition is a consequence of the assumed continuity. If the domain D is such that $A_1, \cdots, A_n$, are negatively infinite while $B_1, \cdots, B_n$ are positively infinite, then Theorem I becomes the following "interval" type theorem:

THEOREM II. *Let the functions $f_i(x, y_1, \cdots, y_n)$ be continuous in $(y_1, \cdots, y_n)$ for each fixed x on (a, b) and all real $(y_1, \cdots, y_n)$ and let these functions be measurable in x on (a, b) for any fixed $(y_1, \cdots, y_n)$ and, finally, let a Lebesgue integrable function $M(x)$ exist such that*

$$|f_i(x, y_1, \cdots, y_n)| < M(x), \qquad i = 1, \cdots, n,$$

for all x on (a, b) and all real $(y_1, \cdots, y_n)$. If $x = c$ is a point of (a, b) and $k_1, \cdots, k_n$ are real numbers, there exists at least one solution of system (1) *on (a, b) such that $y_i(c) = k_i$ $(i = 1, 2, \cdots, n)$.*

Through use of *over* and *under* functions, Perron and others have obtained "interval" type existence theorems[3] which differ materially

[2] *Vorlesungen über reelle Funktionen*, Leipzig, 1928, p. 672.

[3] For discussions of these theorems, see W. M. Whyburn, University of California Publications in Mathematical and Physical Sciences, vol. 1 (1935), pp. 115–134.

in hypotheses from those stated above and which frequently apply when Theorems I and II fail.

The hypotheses of Theorem II exclude linear systems. Separate existence theorems may be proved for such systems or Theorem II may be modified in such a way that it includes them as special cases. The latter result[4] is obtained by replacing $f_i(x, y_1, \cdots, y_n)$ with $F_i(x, y_1, \cdots, y_n)+\sum_{j=1}^{n}G_{ij}(x, y_1, \cdots, y_n)y_j$ and requiring that F_i, G_{ij}, satisfy the same hypotheses originally stated for $f_i(x, y_1, \cdots, y_n)$.

Niccoletti[5] studied a single differential equation of the nth order together with initial conditions at more than one point of the interval. The actual number of such points did not exceed n, the order of the differential equation. Since this work seems to have been the first on differential systems with boundary conditions at more than two points, we indicate the results obtained and the nature of the methods used. Through the use of certain algebraic formulas, Niccoletti established the fact that the function

$$u = \frac{1}{(n-1)!}f(x)\sum_{i=1}^{k}\frac{\partial^{\alpha_i-1}}{\partial a_{1i}^{\alpha_i-1}}\left\{\frac{\alpha_i}{f^{(\alpha_i)}(a_i)(x-a_i)}\int_{a_i}^{x}(a_i-z)^{n-1}\phi(z)dz\right\},$$

where $f(x)=(x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\cdots(x-a_k)^{\alpha_k}$, $\alpha_1+\alpha_2+\cdots+\alpha_k=n$, vanishes along with its first (α_i-1) derivatives at the point $x=a_i$ $(i=1, 2, \cdots, k)$. Furthermore $u^{(n)}=\phi(x)$. If $P_{n-1}(x)$ is a polynomial of degree $n-1$ which together with its first α_i-1 derivatives takes assigned values at $x=a_i$ $(i=1, 2, \cdots, k)$, then $y(x)=u(x)+P_{n-1}(x)$ is a function which satisfies the differential equation $y^{(n)}=\phi(x)$ together with boundary conditions which assign the values of $y(x)$ and its first α_i-1 derivatives at the points $x=a_i$ $(i=1, 2, \cdots, k)$. With this result and the Picard method of successive approximations, Niccoletti proved the following theorem:

THEOREM III. *In the differential equation*

$$y^{(n)} = \phi(x, y, y', \cdots, y^{(n-1)}) \tag{2}$$

let the function $\phi(x, y, y', \cdots, y^{(n-1)})$ be continuous for all x on (a, b) and all real $(y, y', \cdots, y^{(n-1)})$ and let it satisfy a Lipschitz condition with respect to $(y, y', \cdots, y^{(n-1)})$, then there exists a unique solution of (2) which together with its first α_i derivatives takes on assigned values at $x=a_i$ $(i=1, \cdots, k)$, provided the points $a_1, a_2, \cdots, a_k$ all lie on a single subinterval of (a, b) that is sufficiently small.

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[4] See W. M. Whyburn, Annals of Mathematics, (2), vol. 30 (1928), pp. 31–38.

[5] Atti della R. Accademie Delle Scienze Di Torino, vol. 33 (1897), pp. 746 ff.

Niccoletti indicated that his results could be extended to systems of differential equations and to partial differential equations, but it seems clear that the actual extension to systems was not accomplished. Otherwise, the restriction that $y, y', \cdots, y^{(\alpha-1)}$ be assigned at any point where $y^{(\alpha)}$ is given would have been unnecessary. Niccoletti stated that his solutions were unique and that they were continuous functions of the initial values.

Independently of the work of Niccoletti, W. M. Whyburn[6] established a theorem of the type of Theorem III for systems (1) with initial conditions at n or less points of the interval. This theorem requires that the points lie in a single subinterval of (a, b) of length equal to a definite positive number but establishes existence and uniqueness of the solution on the entire interval (a, b). The solution functions are continuous functions of the initial points and values. The essential character of the requirement that the points used in the initial conditions be contained in a restricted subinterval of (a, b) is shown by the example $y' = kz, z' = -ky, y(a_1) = \alpha, z(a_2) = \beta, \alpha^2 + \beta^2 \neq 0$. A solution exists here if $|a_1 - a_2| < \pi/2k$ but fails to exist when $a_2 - a_1 = \pi/2k$.

Theorems of the above type are important since they allow the study of differential systems under conditions which do not permit the simultaneous assignment of all of the initial values but which do admit this assignment over a relatively short range of the independent variable. One possible application would be to a study of rapid machine gun fire where the gun is moved about during the firing interval and the individual trajectories are not as important as the family of such paths generated over a relatively short interval of time. Here one would have a system of second order equations—in number equal to the number of times the gun is fired—together with $2n$ initial conditions assigned at the times $t_1, t_2, \cdots, t_n$, when the gun actually fires. The above theorems, together with approximation methods which grow out of them, would apply to such problems.

This section of the paper is closed with a brief consideration of linear systems in which the notation used in the remainder of the paper is indicated. With the adaptation mentioned above, fundamental existence theorems apply to a system of first order linear differential equations with coefficients which are Lebesgue integrable functions of x on (a, b). It is convenient to write these systems in matrix form[7]

[6] Annals of Mathematics, (2), vol. 30 (1928), pp. 31–38.

[7] This notation has been used by many writers and is becoming quite standard in the literature. One of many references to its use is Birkhoff and Langer, Proceedings of the American Academy of Arts and Sciences, vol. 57 (1922), pp. 51–128.

$$L(Y) = Y' + P(x)Y = Q(x), \tag{3}$$

where $L(Y)$, P, Y, Q, and Y' are square matrices of n-rows. If $x=s$ is a point of (a, b) and if initial conditions are given at this point, the method of variation of parameters yields

$$Y(x) = U(x)\left[\int_s^x V(t)Q(t)dt + V(s)Y(s)\right], \tag{4}$$

where $U(x)$ is a nonsingular matrix solution of the homogeneous equation $L(Y)=0$ and $V(t)=U^{-1}(t)$ is the inverse of $U(t)$. This formula is useful in the remainder of the paper and is important in its own right since it is an explicit expression for the general solution of equation (3) in terms of a solution of the homogeneous equation and the initial values at one point. It is especially useful in a study of the solution of (3) as a *functional* of the coefficients and initial values where it provides estimates of the effects on the solution produced by small variations in the coefficients of the system.[8]

2. **Boundary conditions at two points.** Extensive investigations have been made of system (1) with conditions at two points of the interval. By far the greater part of these studies have been devoted to second order linear systems with parameters in the coefficients. The present paper omits[9] discussion of these results for second order systems and mentions only one development for nth order linear systems—namely that of the Green's function[10] or matrix—which is of interest in connection with later parts of the paper.

Let system (3) be given along with boundary conditions

$$H(Y) = AY(a) + BY(b) = C \tag{5}$$

where A, B, C are nth order square matrices of constants. If the homogeneous system $L(Y)=0$, $H(Y)=0$, is incompatible, then the existence and properties of the Green's matrix may be deduced in the following manner. In formula (4) let $s=a$ and multiply the resulting equation on the left by $AU(a)V(x)$. Similarly, replace s in formula (4) by b and multiply the resulting equation on the left by $BU(b)V(x)$. Addition of these two equations followed by multiplication on the left by $U(x)H^{-1}(U)$ yields

$$Y(x) = U(x)H^{-1}(U)C + \int_a^b G(x, t)Q(t)dt, \tag{6}$$

[8] See W. M. Whyburn, Transactions of this Society, vol. 32 (1930), pp. 502–508.

[9] For references on this literature, see Reid, this Bulletin, vol. 43 (1937), pp. 633–666.

[10] See Bounitsky, Journal de Mathématiques, vol. 5 (1909), pp. 65–125.

where

$$(7) \qquad G(x, t) = \begin{cases} U(x)H^1(U)AU(a)V(t) & \text{for } t < x \\ -U(x)H^1(U)BU(b)V(t) & \text{for } t > x. \end{cases}$$

$G(x, t)$ as given by (7) is the Green's matrix for the system and its properties of uniqueness, continuity, symmetry, and so on, are readily deduced from this explicit formula.

Our discussion of two point boundary conditions is concluded with the observation that, in general, the conditions apply to the end points of the interval under consideration and hence at the boundary of the simply-connected domain (interval) of definition. Cases where one or both of the points are interior to the interval are usually treated by studying the solution on the interval bounded by these points and extending this work to the larger interval through use of results for boundary conditions at one point.

3. **Conditions at a finite number of points.** In this section we consider nth order linear systems with boundary conditions which involve a finite number k of points of the interval. For the differential system (3) the boundary conditions have the form

$$(8) \qquad H(Y) = \sum_{j=1}^{k} A_j Y(a_j) = C,$$

where $A_1, \cdots, A_k$, C, are constant matrices and $a_1 < a_2 < \cdots < a_k$ are k distinct points of (a, b). For the single linear differential equation of the nth order, the boundary conditions are linear combinations of the values of the solution and its first $(n-1)$ derivatives at the k points. M. Picone[11] seems to have been the first person to formulate boundary conditions of type. He also obtained some results for a special nth order differential equation with integral boundary conditions. Bôcher discussed such conditions in his Fifth International Congress address[12] of 1912 and in his Paris lectures[13] of 1913–1914. Bôcher indicated the importance of boundary value problems of this type but did not publish any investigations on them. The first systematic study of such problems was made by C. E. Wilder,[14] who treated the single nth order equation with conditions at k points. Wilder defined a

[11] Rendiconti della Accademia dei Lincei, vol. 17 (1908), pp. 340–347.

[12] Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 1, pp. 163–195.

[13] Bôcher, *Leçons sur les Methodes de Sturm*, Paris, 1917, p. 18.

[14] Transactions of this Society, vol. 18 (1917), pp. 415–442, and vol. 19 (1918), pp. 157–166.

Green's function for his differential system and investigated the adjoint relationship. He then introduced a parameter into one of the coefficients of the equation and developed expansion theorems for the resulting system making use of the Green's function and methods first introduced by Birkhoff.[15] Existence and oscillation theorems for second order systems containing a parameter in a general fashion and with boundary conditions, one of which involved k points, were obtained by W. M. Whyburn.[16] The results here differed from those ordinarily obtained for two point conditions in that the characteristic numbers were not isolated but occurred in sets or in bands. This, in itself, is of interest since the occurrence of bands of characteristic numbers in spectral analysis is frequent. Toyada[17] studied the Green's function and adjoint relationship for nth order differential systems, apparently overlooking the earlier work of Wilder. His results are quite similar to those obtained by Wilder although his method of attack differed in that he did not raise the dimensionality of the space or increase the order of the differential system. Toyada used the skew-symmetry of the Green's matrix as a basis for building an adjoint system and arrived at such a system under the condition that a total of kn^2 identities hold. He attempted to generalize his results to systems with integral boundary conditions but seems to have overlooked difficulties that arise due to the above-mentioned identities. Wilder's previous results showed that, in general, an adjoint system in the ordinary sense would not exist. This result has recently been confirmed by Mansfield.[18]

The connections between k-point boundary value problems and the calculus of variations have been explored by Reid,[19] Denbow,[20] Smiley[21] and Mansfield.[18] This work starts from the relation between the Jacobi necessary condition on the one hand, and second order differential equations with boundary conditions at two points on the other hand, and proceeds to cases where more than two points appear in the boundary conditions. An important device used is a transformation due to Reid[19] which effectively reduces the k-point case to

[15] Transactions of this Society, vol. 9 (1908), pp. 373–395.

[16] Transactions of the Society, vol. 30 (1928), pp. 630–640.

[17] Tohôku Mathematical Journal, vol. 38 (1933), pp. 343–355, and vol. 39 (1934), pp. 387–398.

[18] *Contributions to the Calculus of Variations*, vol. 4, University of Chicago Press, pp. 413–472.

[19] American Journal of Mathematics, vol. 57 (1935), pp. 69–93.

[20] *Contributions to the Calculus of Variations*, 1933–1937, Chicago, 1937, pp. 449–494.

[21] Ibid., pp. 527–566.

one that involves $k-1$ systems each of the two point type. Mansfield[18] made a systematic study of k-point boundary value problems through use of this device and applied his results to the calculus of variations and to differential systems in the complex domain.

Much of the work mentioned above makes specific use of the fact that the number of points appearing in the boundary conditions is finite. In some cases, use is made of an increase in the dimensionality of the space while in other cases, the order of the differential system is raised. When it is observed that the introduction of interior points of the interval into the boundary conditions has the effect of changing the simply-connected domain of definition into one that is multiply-connected, it is not surprising that devices of the above-mentioned type are used. However, difficulties arise when attempts are made to generalize the differential system through having the boundary conditions contain integrals or infinitely many points. The next two sections discuss conditions of these types.

4. **Integral conditions.** When the number of points which enter the boundary conditions is no longer required to be finite, there are possibilities that the conditions will contain linear combinations of the solution values on a non-dense infinite point set or contain definite integrals of the solution over the interval. Picone[22] investigated relationships that exist between integral equations and certain special nth order differential equations with integral boundary conditions. von Mises[23] studied the special second order system

$$(9)\qquad \begin{cases} \dfrac{d}{dx}\left[k\,\dfrac{du}{dx}\right] + (\lambda g - q)u = 0 \\ \displaystyle\int_a^b A(x)u(x)dx = 0, \qquad \int_a^b B(x)u(x)dx = 0, \end{cases}$$

where k, g, q, A, B are continuous functions of x and λ is a parameter. von Mises was interested in this system because of its occurrence in certain problems of hydrodynamics.[24] His special case of hydrodynamic application had $k=q=1$, $g=x+c$, $A=1/B=e^x$, where c is a constant. With specializations in the direction of this application, von Mises used Sturm's method of passage to the limit from an algebraic system to obtain existence and oscillation theorems for system (9).

[22] Loc. cit., pp. 340–347.

[23] *Festschrift Heinrich Weber*, Leipzig, 1912, pp. 252–282.

[24] See Sommerfeld, Proceedings of the Fourth International Congress of Mathematicians, Rome, 1909, vol. 3, p. 116.

Bôcher[25] mentioned integral boundary conditions for second order systems and indicated a transformation that could sometimes be applied to replace such conditions by conditions at two points. The transformation suggested by Bôcher depends upon the actual solution in closed form of a slightly modified form of the given differential equation and, for this reason, offers little probability of use in any particular problem. Bôcher observed that this transformation would replace system (9) by one which no longer involved the parameter in the simple linear fashion of its original occurrence. In general, systems with integral boundary conditions should be regarded as essentially different from those with two point or k-point conditions. It is clear that integral boundary conditions would arise quite naturally in hydrodynamics or indeed in any investigation where such things as fixed areas under solution curves, fixed lengths of curves, and so on enter.

Tamarkin[26] considered k-point and integral conditions for single nth order linear differential equations.

W. M. Whyburn[27] established existence and oscillation theorems for linear second order equations with one integral boundary condition and one condition of the two point type. The coefficients of the system contained a parameter in a general (non-linear) fashion. As in the case of k-point conditions, it was found that the characteristic numbers were not isolated but occurred in sets which might be continua. Toyada[28] set up a Green's matrix for nth order linear systems with integral boundary conditions and attempted to generalize the adjoint notion to systems of this type. His methods were extensions of ones that he had used for k-point conditions and such extensions were possible because these methods were largely independent of the number of points involved in the boundary conditions. In connection with his work on adjoint systems, Toyada imposed a set of conditions the number of which depended upon the number of points involved in the boundary conditions. These conditions become unruly when integral conditions are used.

Integral boundary conditions of a different type have been investigated by a number of people.[29] These conditions contain Stieltjes integrals and for the nth order linear system (3) have the form

[25] Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 1, p. 167.

[26] *On Some General Problems of the Theory*, (in Russian), Petrograd, 1917.

[27] Transactions of this Society, vol. 30 (1928), pp. 630–640.

[28] Loc. cit., p. 391.

[29] See N. Cioranescu, Bulletinul, Facultătea de Stiinte din Cernăuti, vol. 5 (1931), p. 99 and Mathematische Zeitschrift, vol. 35 (1932), pp. 601–608. Also see A. Smorgorshewsky, Recueil Mathématique, (n.s.), vol. 7 (1940), pp. 179–196.

$$H(Y) = \int_a^b dF(x)Y(x) = C, \tag{10}$$

where $H(Y)$, C, $F(x)$, Y are square matrices of n-rows and the elements of $F(x)$ are of bounded variation on (a, b). Boundary conditions of this type are quite general and contain all of the conditions discussed above as special cases. For example, conditions (8) involving k points of the interval are obtained when $F(x)$ is defined as a matrix of step functions—constant on each of the subintervals $a_i < x < a_{i+1}$ but with discontinuities at the points $x = a_i$ given by $F(a_i^+) - F(a_i^-) = A_i$ $(i = 1, \cdots, n)$, where $F(a_1^-) = F(a)$, $F(a_k^+) = F(b)$. In this case

$$\int_a^b dF(x)Y(x) = \sum_{i=1}^{k} A_i Y(a_i).$$

Integral boundary conditions of the type $\int_a^b F(x)\,Y(x)dx = C$ are reduced to the above type when use is made of the relations between Stieltjes integrals and those of Riemann or Lebesgue types. Smorgorshewsky[30] developed the notions of Green's matrix and generalized Green's matrix for systems of type (3), (10), and investigated the properties of these matrices. The fact that linear boundary conditions of quite general type can be replaced by conditions containing Stieltjes integrals has been known for many years. However, partly due to the somewhat unwieldly character of Stieltjes integrals, few advantages have resulted from such replacement. The above-mentioned results for the Green's matrix represent one place, at least, where advances have been made through use of Stieltjes integrals.

5. **Boundary conditions at infinitely many points.** In this section nth order differential systems with boundary conditions at infinitely many points of the interval are discussed. The most general sets considered are those of the *first species*. If g is a point set of the first species and if $A_1, A_2, \cdots$ is an absolutely convergent[31] sequence of constant matrices, the boundary conditions for system (3) are of the form

$$H_1(Y) = \sum_{1}^{\infty} A_i Y(d_i) = C_1, \tag{11}$$

where $d_1, d_2, \cdots$ is some ordering of the elements of g into a countable sequence. It is a consequence of the absolute convergence of $\sum_1^\infty A_i$ that conditions (11) are equivalent to those obtained by any

[30] Loc. cit., pp. 179–196.

[31] That is to say, each element sequence converges absolutely.

other countable ordering of the elements of g. More general boundary conditions are obtained if an integral term occurs in (11) as follows:

$$H_2(Y) = \sum_1^\infty A_i Y(d_i) + \int_a^b F_2(x)Y(x)dx = C_2. \tag{12}$$

In a forthcoming article [22] the present author has shown that differential system (3) together with boundary conditions (12) is equivalent to a system consisting of (3) and conditions

$$H(Y) = AY(a) + BY(b) + \int_a^b F(x)Y(x)dx = C. \tag{13}$$

These systems are equivalent in the sense that any solution of the one system is also a solution of the other. Details of the proof are omitted here since they appear in the article cited. It is to be noted that the matrix function $F(x)$ is Lebesgue integrable on (a, b) provided $F_2(x)$ has this property. In general, $F(x)$ is not continuous and may fail to be Riemann integrable. A further formal reduction of boundary conditions (13) is accomplished if $F(x)$ is assumed to have properties that permit replacement of the integral and two point terms by a single Stieltjes integral. However, this replacement does not actually simplify the system and does lose certain advantages that result from the form of conditions (13). To illustrate these advantages, we develop the notion of Green's matrix for system (3), (13) and thus for system (3), (12).

Let $U(x)$ and $V(x)$ have the same meanings as in formula (4) and let $Y(x)$ denote the general solution of equation (3). Multiplication of equation (4) on the left by $F(s)U(s)V(x)ds$ followed by integration, once between the limits $s=a$ and $s=x$ and once between the limits $s=b$ and $s=x$ yields

$$\int_a^x F(s)U(s)V(x)Y(x)ds = \int_a^x F(s)Y(s)ds + \int_a^x ds \int_s^x F(s)U(s)V(t)Q(t)dt, \tag{14}$$

$$\int_b^x F(s)U(s)V(x)Y(x)ds = \int_b^x F(s)Y(s)ds + \int_b^x ds \int_s^x F(s)U(s)V(t)Q(t)dt. \tag{15}$$

[22] Soon to appear in University of California Publications in Mathematics, University of California Press, Los Angeles.

When the double integrals in (14) and (15) are integrated by parts and the function

$$G_2(x, t) = \begin{cases} \int_a^t F(s)U(s)V(t)ds & \text{for } t < x \\ \int_b^t F(s)U(s)V(t)ds & \text{for } t > x \end{cases}$$

is introduced, equation (14) and (15) combine to yield

$$\text{(16)} \quad \int_a^b F(t)U(t)V(x)Y(x)dt = \int_a^b F(t)Y(t)dt + \int_a^b G_2(x, t)Q(t)dt.$$

Let $s=a$ in equation (4) and multiply this equation on the left by $AU(a)V(x)$. Similarly, let $s=b$ in equation (4) and multiply this equation on the left by $BU(b)V(x)$. Addition of the two resulting equations to equation (16) yields

$$\text{(17)} \quad H(U)V(x)Y(x) = H(Y) + \int_a^b [G_1(x, t) + G_2(x, t)]Q(t)dt,$$

where

$$G_1(x, t) = \begin{cases} AU(a)V(t) & \text{for } t < x \\ -BU(b)V(t) & \text{for } t > x. \end{cases}$$

If the homogeneous system $L(U)=0$, $H(U)=0$ is assumed incompatible, then $H(U)^I$ exists and equation (17) yields the unique solution of system (3), (13) in the form

$$\text{(18)} \quad Y(x) = U(x)H(U)^IC + \int_a^b G(x, t)Q(t)dt,$$

where the Green's matrix $G(x, t)$ is given explicitly by

$$\text{(19)} \quad G(x, t) = \begin{cases} U(x)H(U)^I\left[AU(a) + \int_a^t F(s)U(s)ds\right]V(t) & \text{for } t < x \\ -U(x)H(U)^I\left[BU(b) + \int_t^b F(s)U(s)ds\right]V(t) & \text{for } t > x. \end{cases}$$

$G(x, t)$ as defined by equation (19) has the usual properties of continuity, uniqueness, and so on, of the Green's matrix for systems with less general boundary conditions. If the function $F(x)$ is taken as the zero matrix, the differential system (3), (13) becomes the two point system (3), (5) and formula (19) shows immediately that $G(x, t)$ be-

comes the Green's matrix defined by equation (7). Examination of formula (19) reveals the fact that symmetry of $G(x, t)$ in x and t is to be expected only in the degenerate two point case $F(x) \equiv 0$.

Although an adjoint system for (3), (13) in the ordinary sense does not exist in general, it is possible to set up boundary conditions similar to those given in (13) which can be taken with the adjoint equation

$$M(Z) = Z' - Z(x)P(x) = 0 \tag{20}$$

to form a very useful system associated with (3), (13). When $F(x) \equiv 0$ this system becomes the adjoint system for (3), (13).

Simultaneous study of the Green's matrices for (3), (13) and this associated system shows that they satisfy relations which readily degenerate into the well known symmetry relations for the case of two point systems.

6. **Conclusion.** This paper has attempted to review a body of literature on boundary value problems for ordinary differential equations which has not, to our knowledge, been summarized before. Although many of the results cited have appeared in print, it is believed that the paper is not without some novelty. It is hoped that presentation of the paper at this time will lead to applications of systems with more general boundary conditions in a number of fields of applied mathematics. The author does not believe that known results for such systems are adequate to handle many of these applications, but he does think that additional results can be obtained as needs for them arise in applied problems. It is also likely that many problems now treated as two point boundary value problems would receive better treatment if they were studied through the use of more general boundary conditions.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

ON THE LOGARITHMIC MEANS OF REARRANGED PARTIAL SUMS OF A FOURIER SERIES

OTTO SZÁSZ

Let $f(\theta)$ be a real, even and Lebesgue integrable function; let

$$f(\theta) \sim (1/2)a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta.$$

We write

$$s_0 = (1/2)a_0, \qquad s_n = (1/2)a_0 + a_1 + \cdots + a_n, \qquad n \geqq 1,$$

and denote by $s_0^*, s_1^*, \cdots, s_n^*$ the values of $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order. In 1935 Hardy and Littlewood [2][1] proved the following remarkable theorem:

THEOREM 1. *If*

$$f(\theta) = o\left(\log \frac{1}{\theta}\right)^{-1} \tag{1}$$

for small positive θ, then

$$\sum_{0}^{n} \frac{s_v^*}{v+1} = o(\log n). \tag{2}$$

Hardy and Littlewood gave two applications of this theorem by proving:

THEOREM 2. *If* (1) *holds, then*

$$\sum_{1}^{n} |s_v|^q = o(n) \tag{3}$$

for every positive q.

THEOREM 3. *If* (1) *holds and if*

$$a_n > -An^{-\xi} \tag{4}$$

for a positive A and ξ, then $s_n \to 0$.

They have also proved [1, Theorem 9] that in Theorem 3 the

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

assumption (1) can be replaced by

$$F(\theta) \equiv \int_0^{\theta} |f(t)|\,dt = o\left(\frac{\theta}{\log(1/\theta)}\right), \tag{5}$$

but the proof requires a very difficult Tauberian theorem. Thus the question arose whether Theorem 1 remains true when (1) is replaced by (5).

In partial answer to this question we shall prove the following result.

THEOREM I. *If* (5) *holds and if for small positive* θ

$$|f(\theta)| < \theta^{-c}, \qquad c \text{ a positive constant}, \tag{6}$$

then (2) *holds.*

Note that condition (6) permits $f(\theta)$ to become as large as any power of $1/\theta$ near $\theta = 0$.

For the proof of Theorem I we follow the device of Hardy and Littlewood. We suppose that $|s_m| = s_{v_m}^*$, $0 \leqq m \leqq n$, so that $v_0, v_1, \cdots, v_n$ is a permutation of $0, 1, \cdots, n$. We then have

$$S_n \equiv \sum_0^n \frac{s_m^*}{m+1} = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta)}{2\sin\theta/2}\, g(\theta)\,d\theta,$$

where

$$g(\theta) = \sum_0^n \frac{\operatorname{sgn} s_m}{v_m + 1} \sin(m + 1/2)\theta.$$

We choose δ so that (6) holds for $0 < \theta < \delta$, and write

$$\begin{aligned} S_n &= \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right)\frac{f(\theta)}{\sin\theta/2}\, g(\theta)\,d\theta \\ &\equiv I_1 + I_2 + I_3, \qquad\qquad \delta < 1 < n. \end{aligned}$$

We first estimate I_1:

$$\begin{aligned} |I_1| &\leqq \int_0^{1/n} \frac{|f(\theta)|}{\theta}\,|g(\theta)|\,d\theta \leqq \int_0^{1/n} |f(\theta)|\left(\sum_0^n \frac{m+1/2}{v_m+1}\right)d\theta \\ &\leqq (n+1/2)\left(\sum_{v=0}^{n} \frac{1}{v+1}\right)\int_0^{1/n} |f(\theta)|\,d\theta; \end{aligned} \tag{7}$$

hence, by (5)

$$I_1 = o(1) \qquad \text{as } n \to \infty.$$

Next

$$|I_3| = \left| \sum_{0}^{n} \frac{\operatorname{sgn} s_m}{v_m + 1} j_m \right|,$$

where

$$j_m = \frac{1}{\pi} \int_{\delta}^{\pi} \frac{f(\theta)}{\sin \theta/2} \sin (m + 1/2)\theta d\theta.$$

To any given $\epsilon > 0$ we can choose, by the Riemann-Lebesgue theorem, $m_0(\epsilon)$ so that $|j_m| < \epsilon$ for $m \geqq m_0$. Then

$$\begin{aligned} |I_3| &< \frac{1}{\pi} \frac{1}{\sin \delta/2} \int_{\delta}^{\pi} |f(\theta)|\, d\theta \cdot \sum_{0}^{m_0} \frac{1}{v_m + 1} + \epsilon \sum_{m_0}^{n} \frac{1}{v_m + 1} \\ &< \frac{m_0 + 1}{\pi \sin \delta/2} \int_{\delta}^{\pi} |f(\theta)|\, d\theta + \epsilon \log (n + 1), \end{aligned} \tag{8}$$

hence $I_3 = o(\log n)$, as $n \to \infty$.

Theorem I is now reduced to showing that $I_2 = o(\log n)$. We have from Young's inequality [2, p. 319]

$$uv \leqq lu \log^+ u + le^{(v-1)/l}$$

for $u > 0$, $l > 0$, and all real v. Here $\log^+ u = \max(0, \log u)$. We take $u = |f(\theta)/\sin \theta/2|$, $v = |g(\theta)|$, $l = 2$; then

$$\begin{aligned} |I_2| &< \frac{2}{\pi} \int_{1/n}^{\delta} \left| \frac{f(\theta)}{\sin \theta/2} \right| \log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| d\theta + \frac{1}{\pi} \int_{1/n}^{\delta} e^{|g(\theta)|/2} d\theta \\ &= K + L, \qquad \text{say.} \end{aligned}$$

Now from (6)

$$\log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| < \log \pi \theta^{-c-1} < (c + 1) \log \frac{\pi}{\theta},$$

hence

$$K \leqq 2(c + 1) \int_{1/n}^{\delta} |f(\theta)|\, \theta^{-1} \log \frac{\pi}{\theta} d\theta.$$

Integration by parts yields

$$K \leqq 2(c + 1) \left[F(\delta)\delta^{-1} \log \frac{\pi}{\delta} + \int_{1/n}^{\delta} F(\theta)\theta^{-2} \left\{ \log \frac{\pi}{\theta} + 1 \right\} d\theta \right], \tag{9}$$

and (5) gives $K = o(\log n)$.

Finally, to estimate L, we use the following two lemmas (cf. Hardy and Littlewood [2]):

LEMMA A. *If*

$$\phi(\theta) = \sum_{-n}^{n} c_\nu e^{i\nu\theta}, \qquad \phi^+(\theta) = \sum_{-n}^{n} c_\nu^+ e^{i\nu\theta},$$

where c_ν^+ *are the numbers* $|c_\nu|$ *rearranged so that* $c_0^+ \geqq c_{-1}^+ \geqq c_1^+ \geqq c_{-2}^+ \geqq \cdots$; *then*

$$\int_{-\pi}^{\pi} e^{b|\phi(\theta)|}d\theta \leqq 2\int_{-\pi}^{\pi} e^{b|\phi^+(\theta)|}d\theta, \qquad \text{for every } b > 0.$$

LEMMA B. $\left|\sum_1^n \cos \nu\theta/(\nu+1)\right| < \log(1/|\theta|)+c$ *for all* θ *and* n, *and a constant* $c>0$.

Now, if we put

$$\frac{\operatorname{sgn} s_m}{\nu_m + 1} = \rho_m,$$

we have

$$g(\theta) = \Im(e^{i\theta/2}\gamma(\theta)), \qquad |g(\theta)| \leqq |\gamma(\theta)|,$$

where

$$\gamma(\theta) = \sum_0^n \rho_m e^{im\theta}$$

is a polynomial of degree n whose coefficients have (in some order) the absolute values $1, 1/2, \cdots, 1/(n+1)$. Hence, in the notation of the above lemma.

$$\gamma^+(\theta) = 1 + e^{-i\theta}/2 + e^{i\theta}/3 + e^{-2i\theta}/4 + \cdots,$$

the last term having modulus $1/(n+1)$ and an argument depending on the parity of n. The imaginary part of $\gamma^+(\theta)$ is bounded in n and θ, and the modulus of the real part is less than $\log(1/|\theta|)+c$, c a positive constant, due to Lemma B; hence $|\gamma^+(\theta)| < \log(1/|\theta|)+c$. We now obtain

$$(10) \qquad L < \frac{1}{\pi}\int_0^{\pi} e^{|\gamma(\theta)|/2}d\theta < \int_0^{\pi} e^{|\gamma^+(\theta)|/2}d\theta < \int_0^{\pi} \theta^{-1/2}d\theta + c;$$

the theorem now follows from (7), (8), (9), and (10).

THEOREM II. *To a given* $\delta>0$ *denote by* $\nu(\delta, n)$ *the number of values* m *for which* $m \leqq n$, $|s_m| > \delta$; *then* $\log \nu = o(\log n)$ *as* $n \to \infty$.

We have

$$\delta \log \nu < \sum_{0}^{n} \frac{s_m^*}{m+1} = o(\log n), \qquad \text{from (2);}$$

δ being fixed our theorem follows [2, Theorem 3].

THEOREM III. *If* (5) *holds then* $s_n = o(\log \log n)$.

The proof is straightforward, using

$$s_n = \frac{1}{\pi} \int_0^{\pi} \frac{f(\theta) \sin (n+1/2)\theta}{\sin \theta/2} d\theta.$$

We have

$$|s_n| < \int_0^{\pi} \frac{|f(\theta)|\,|\sin (n+1/2)\theta|}{\theta} d\theta = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}$$
$$\equiv A_1 + A_2 + A_3,$$

where we choose $\delta = \delta(\epsilon)$ so that

$$F(\theta) < \frac{\epsilon\theta}{\log 1/\theta} \qquad \text{for } 0 < \theta < \delta.$$

Now $A_1 < (n+1/2)F(1/n) = o(1)$ as $n \to \infty$;

$$A_2 < \int_{1/n}^{\delta} |f(\theta)|\, \theta^{-1} d\theta = F(\theta)\cdot\theta^{-1}\Big]_{1/n}^{\delta} + \int_{1/n}^{\delta} F(\theta)\cdot\theta^{-2} d\theta$$
$$< \frac{\epsilon}{\log 1/\delta} + \epsilon \int_{1/n}^{\delta} \frac{d\theta}{\theta \log 1/\theta} = \frac{\epsilon}{\log 1/\delta} + \epsilon \log \log \frac{1}{\theta}\Big]_{1/n}^{\delta}$$
$$< \epsilon/\log 1/\delta + \epsilon \log \log n.$$

Hence for fixed ϵ and δ

$$\limsup_{n\to\infty} \frac{A_2}{\log \log n} \leqq \epsilon;$$

but ϵ is arbitrarily small, thus $A_2 = o(\log \log n)$. Finally

$$A_3 < \delta^{-1} \int_0^{\pi} |f(\theta)|\, d\theta = o(\log \log n);$$

this proves Theorem III.

THEOREM IV. *Denote by* $\phi(x)$ *a function satisfying the conditions* $\phi(0) = 0$, $\phi(x) \uparrow$, *as* $x \uparrow$; $\phi(x) < \exp(e^{cx})$ *for all* $x > 0$; *c a positive constant. Under the assumptions of Theorem* I, $\sum_1^n \phi(|s_m|) = o(n)$. *We may say the sequence* $\{s_n\}$ *is strongly summable relative to* $\phi(x)$.

We write for a given $\delta>0$

$$\sum_1^n \phi(|s_m|) = \sum_{|s_m|<\delta} + \sum_{|s_m|\geq\delta} \equiv B_1 + B_2;$$

then $B_1<n\phi(\delta)$. By Theorems II and III we may write

$$|s_n| = \epsilon_n \log\log n, \quad \log\nu = \eta_n \log n, \qquad \text{where } \epsilon_n \to 0,\ \eta_n \to 0.$$

Now

$$B_2 < \nu\phi(\epsilon_n \log\log n) < n^{\eta_n}\cdot\exp\left\{(\log n)^{\epsilon_n}\right\} = o(n),$$

hence

$$\limsup_{n\to\infty} \frac{1}{n}\sum_1^n \phi(|s_m|) \leqq \phi(\delta),$$

letting $\delta\to 0$ our theorem follows.

THEOREM V. *The assumptions* (4), (5), *and* (6) *imply* $s_n\to 0$.

This follows from Theorem II; see the proof of Theorem 5 in [2].

Final remark. In the more general case, when

$$f(\theta) \sim (1/2)a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta),$$

and when the point under consideration is θ_0, we use the familiar reduction. We have

$$(1/2)\{f(\theta_0+\theta) + f(\theta_0-\theta)\} \sim (1/2)\alpha_0 + \sum_1^\infty \alpha_n \cos n\theta,$$

where

$$\alpha_n = a_n \cos n\theta_0 + b_n \sin n\theta_0, \qquad n \geqq 1.$$

Suppose there is an s, for which

$$\int_0^\theta |f(\theta_0+t) + f(\theta_0-t) - 2s|\,dt = o\left(\frac{\theta}{\log 1/\theta}\right), \qquad \text{as } \theta\to 0,$$

and

$$|f(\theta_0+\theta) + f(\theta_0-\theta) - 2s| < 2|\theta^{-c}| \quad \text{for small } |\theta|.$$

Then application of Theorem I yields:

$$\sum_0^n \frac{1}{v+1} s_v^*(\theta_0) = o(\log n),$$

where $s_v^*(\theta_0), v=0, 1, \cdots, n$, is the sequence $|s_v(\theta_0)-s|$ in decreasing order.

References

1. G. H. Hardy and I. E. Littlewood, *Some new convergence criteria for Fourier series*, Annali, Scuola Normale Superiore, Pisa, (2), vol. 3 (1934), pp. 43–62.

2. ———, *Notes on the theory of series* (XVIII): *On the convergence of Fourier series*, Proceedings of the Cambridge Philosophical Society, vol. 31 (1935), pp. 317–323.

3. A. Zygmund, *Trigonometrical Series*, 1935.

University of Cincinnati

THE BASIC ANALOGUE OF KUMMER'S THEOREM[1]

J. A. DAUM

1. **Introduction.** About one hundred years ago, E. E. Kummer[2] proved the formula

$$(1)\qquad {}_2F_1\left[\begin{matrix}a, & b; \\ 1+a-b & \end{matrix}\ -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}$$

which has since been known as Kummer's theorem. This appears to be the simplest relation involving a hypergeometric function with argument (-1).

All the relations in the theory of hypergeometric series ${}_rF_s$ which have analogues in the theory of basic series[3] are those in which the argument is $(+1)$. Apparently, there has been no successful attempt to establish the basic analogue of any formula involving a function ${}_rF_s(-1)$. Since Kummer's theorem is fundamental in the proofs of numerous relations between hypergeometric functions of argument (-1), it seemed desirable that an attempt be made to prove the basic analogue of Kummer's theorem and to investigate the possibility of obtaining new relations in basic series with arguments corresponding to the argument (-1) in the classical case.

In this paper, the basic analogue of Kummer's theorem is obtained

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[1] The results presented in this paper are included in a dissertation for the doctorate, University of Nebraska, 1941.

[2] E. E. Kummer, *Ueber die hypergeometrische Reihe*, Journal für die reine und angewandte Mathematik, vol. 15 (1836), pp. 39–83.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, chap. 8.

by a simple specialization of a result due to Bailey.[4] The existence of a basic analogue of Dixon's theorem[5] is discussed and a new type of relation connecting a modified basic series and one of the conventional type is presented.

2. **Derivation of the basic analogue of Kummer's theorem.** In the formula:[4]

$$ {}_6\Phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & d, & e, & f; & aq/def \\ & a^{1/2}, & -a^{1/2}, & aq/d, & aq/e, & aq/f & \end{matrix}\right] = \frac{\Omega(aq/d)\Omega(aq/e)\Omega(aq/f)\Omega(aq/def)}{\Omega(aq)\Omega(aq/ef)\Omega(aq/df)\Omega(aq/de)} $$

where[6]

$$ \Omega(z) = \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-zq^n}, $$

put $e=a^{1/2}$, $f=-a^{1/2}$, and $d=b$, and obtain

$$ {}_2\Phi_1\left[\begin{matrix} a, b; & -q/b \\ aq/b & \end{matrix}\right] = \frac{\Omega(aq/b)\Omega(qa^{1/2})\Omega(-qa^{1/2})\Omega(-q/b)}{\Omega(aq)\Omega(-q)\Omega(qa^{1/2}/b)\Omega(-qa^{1/2}/b)}. $$

This is the basic analogue of Kummer's theorem. It should be noted that it cannot be obtained from the basic analogue of Gauss's theorem.[7]

3. **Application.** The basic analogue of Kummer's theorem immediately suggests the existence of the basic analogue of Dixon's theorem. It has been shown by Watson[8] that Dixon's theorem, which sums a general well poised ${}_3F_2$ with unit argument, can be derived from the theorems of Gauss and Kummer by means of an interchange in the order of summation.

Unfortunately, this method, when applied to the basic analogues, does not lead to an analogue of Dixon's theorem. The difficulty is due to an essential difference in the properties of the omega and

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[4] W. N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quarterly Journal of Mathematics, Oxford, vol. 7 (1936), p. 105.

[5] A. G. Dixon, *Summation of certain series*, Proceedings of the London Mathematical Society, (1), vol. 35 (1903), pp. 285–289.

[6] N. A. Hall, *An algebraic identity*, Journal of the London Mathematical Society, vol. 11 (1936), p. 276.

[7] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 68.

[8] G. N. Watson, *Dixon's theorem on generalized hypergeometric functions*, Proceedings of the London Mathematical Society, (2), vol. 22 (1924), pp. xxxii–xxxiii (records for May 1923). See also W. N. Bailey, *Generalized Hypergeometric Series*, p. 13.

gamma functions. However, a slight modification of this method does lead to a new and curious result, namely,

$$\sum_{n=0}^{\infty} \frac{\Omega(aq^n)\Omega(bq^n)\Omega(cq^n)}{\Omega(q^{n+1})\Omega(aq^{n+1}/b)\Omega(aq^{n+1}/c)} (aq/bc)^n q^{n(n+1)/2} \tag{2}$$
$$= \frac{\Omega(a)\Omega(b)\Omega(c)}{\Omega(aq)\Omega(aq/bc)} {}_3\Phi_2\left[\begin{matrix} b, & c, & -q; \\ qa^{1/2}, & -qa^{1/2} & \end{matrix} \; aq/bc\right].$$

The argument requires that a certain double series be absolutely convergent.

The left member of (2) is a modified basic series somewhat similar to some of the q-hypergeometric series studied by Jackson.[9] Since Dixon's theorem sums a general well-poised ${}_3F_2$ with unit argument, (2) cannot be considered a basic analogue of Dixon's theorem. This result is interesting inasmuch as it expresses what may be called a pseudo-basic series in terms of a function ${}_3\Phi_2$. This seems to be the first result of its type to appear in the theory of basic series.

AGRICULTURAL AND MECHANICAL COLLEGE OF TEXAS AND
UNIVERSITY OF NEBRASKA

[9] F. H. Jackson, *Summation of q-hypergeometric series*, Messenger of Mathematics, vol. 50 (1921), p. 101.

PERRON INTEGRALS

R. L. JEFFERY

At the recent conference on the theory of integration held at the University of Chicago Professor E. J. McShane remarked that formulas for integration by parts and for change of variable in the integrand have been established for the special Denjoy integral, and consequently for the Perron integral, since the two integrals are equivalent. He then raised the question as to the possibility of proving these formulas directly from the Perron definitions. Later, as we were waiting at the station for my train to leave, Professor Graves talked over with me the implications in McShane's point of view. The following solutions are the outcome of this conversation.

Integration by parts. *Let $g(x)$ be a continuous function on the interval (a, b) for which g' is finite except possibly for a denumerable set, and summable. Let the function $f(x)$ be integrable in the Perron sense and set*

$$F(x) = \int_a^x f(x)dx.$$

Then $f(x)g(x)$ is integrable in the Perron sense, and

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx.$$

The conditions on g' make g absolutely continuous, and consequently the difference of two non-decreasing functions each of which is positive or zero. Hence we need consider only the case in which $g(x)$ is non-decreasing and positive or zero. Let $\psi(x)$ be a major function to $f(x)$. Consider $D_*(\psi g)$ which is the lower limit as $h \to 0$ of

$$\frac{\psi(x+h)g(x+h) - \psi(x)g(x)}{h}$$

$$= \frac{g(x+h)\{\psi(x+h) - \psi(x)\}}{h} + \frac{\psi(x)\{g(x+h) - g(x)\}}{h}.$$

From the continuity of g and the fact that both g and g' are not less than zero it follows that

$$D_*(\psi g) \geqq gD_*\psi + \psi g'$$

at every point for which g' exists. Then, since $D_*\psi > -\infty$ except for a

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denumerable set, and $g' \geqq 0$ and finite except for a denumerable set, we have $D_*(\psi g) > -\infty$ except for a denumerable set. Furthermore, from $g' \geqq 0$, $\psi \geqq F$, and $D_*\psi \geqq f$ almost everywhere we get

$$D_*(\psi g) \geqq fg + Fg'$$

almost everywhere. These considerations together with the continuity of ψ and g show that ψg is a major function to $fg+Fg'$. Furthermore, if ϕ is a minor function to f it can be shown in a similar way that ϕg is a minor function to $fg+Fg'$. Again, if the difference $\psi-\phi$ is sufficiently small the difference $\psi g-\phi g$ is arbitrarily small. Then, since the common bound of ψg and ϕg is Fg, we conclude that $fg+Fg'$ is Perron integrable to Fg. The function F is continuous, and g' is summable. Hence Fg' is summable and, a priori, Perron integrable. Then fg being the difference of two Perron integrable functions is itself Perron integrable, and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b Fg'\,dx, \tag{1}$$

which is the desired result.

In the most general formula so far obtained[1] the only restriction on $g(x)$ is that it be of bounded variation. Then fg is Perron integrable and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b F\,dg. \tag{2}$$

In this case the foregoing procedure cannot be used for two reasons: The function ψg is not continuous at the discontinuities of g; if $g=+\infty$ at the points of a more than denumerable set and ψ is negative at the points of this set then $D_*(\psi g) = -\infty$ at the points of this more than denumerable set. Dropping the requirement that g be continuous, we prove the following result.

If $g(x)$ is of bounded variation and g' is finite except for a denumerable set, then formula (2) *holds.*

At the points of discontinuity of g it is necessarily true that g' is infinite, but this set is not more than denumerable. Our restriction is therefore that g' cannot be infinite at a more than denumerable set other than discontinuities. As in the proof of (1) there is no loss of generality in taking $g \geqq 0$ and non-decreasing. Let ψ be a major function to F with $\psi - F < \epsilon$, and consider

[1] Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 201.

$$R(x) = \psi(x)g(x) - F(a)g(a) - \int_a^x \psi dg.$$

Using the law of the mean for Stieltjes integrals we have

$$\begin{aligned} R(x+h) - R(x) &= \psi(x+h)\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \\ &\quad + \psi(\xi)\{g(x+h) - g(x)\} \\ &= \{\psi(x+h) - \psi(\xi)\}\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \end{aligned}$$

where ξ is on the interval defined by the points x and $x+h$. The continuity of R then follows from that of ψ. Again

$$\frac{R(x+h) - R(x)}{h} = \{\psi(x+h) - \psi(\xi)\}\frac{g(x+h) - g(x)}{h} + g(x)\frac{\psi(x+h) - \psi(x)}{h}.$$

The lower limit of the second term on the right is not greater than $g(x)D_*\psi > -\infty$ except for a denumerable set. The lower limit of the first term on the right can be $-\infty$ at points for which $g' = +\infty$ (h and ξ taking values which make $\psi(x+h) - \psi(\xi)$ negative). But $g' = +\infty$ for at most a denumerable set. Hence we have

$$D_*R > -\infty$$

except for a denumerable set. At a point for which g' is finite the limit of the first term on the right is zero. Hence we have

$$D_*R = gD_*\psi \geqq fg$$

almost everywhere. From these considerations it follows that R is a major function to fg. Also from $\psi - F < \epsilon$ we get

$$\theta(x) = \int_a^x \psi dg - \int_a^x F dg = \int_a^x \eta dg$$

where $0 \leqq \eta < \epsilon$. Hence $0 \leqq \theta(x) < \epsilon\{g(b) - g(a)\}$. From this and $\psi \geqq F$ we see that by taking a sequence of values of ϵ tending to zero we get a set of major functions $R(x)$ whose lower bound is

$$F(x)g(x) - F(a)g(a) - \int_a^x F dg.$$

Working in a similar way with functions ϕ which are minor to f we arrive at a set of functions which are minor to fg, and whose upper bound is the same as the lower bound of the set R. It then follows that fg is Perron integrable, and

$$\int_a^b fgdx = [Fg]_a^b - \int_a^b Fdg. \tag{2}$$

Formula (2) includes formula (1), but a separate proof of (1) is of interest because of its simplicity.

Change of variable. *Let $f(x)$ be Perron integrable on (a, b), and let $x(t)$ be a continuous non-decreasing function of t on the interval (α, β) where $a=x(\alpha)$, $b=x(\beta)$, and on this interval let $x'(t)$ be finite except perhaps for a denumerable set. Then the function $f\{x(t)\}x'(t)$ is Perron integrable on this interval (α, β), and*

$$\int_a^b f(x)dx = \int_\alpha^\beta f\{x(t)\}x'(t)dt.$$

Let $\psi(x)$ be a major function to $f(x)$, and let $\chi(t)=\psi\{x(t)\}$. Then $\psi(t)$ is continuous. Also, since $D_*\psi > -\infty$ except for a denumerable set, and $x'(t)\geqq 0$ and finite except for a denumerable set, we have

$$D_*\chi(t) = D_{*x}\psi x'(t) > -\infty$$

except for a denumerable set. Again, since $D_{*x}\psi(x)\geqq f(x)$ almost everywhere, and $x'(t)\geqq 0$, we have

$$D_*\chi(t) \geqq f(x)x'(t) = f\{x(t)\}x'(t)$$

almost everywhere. From these results it follows that $\chi(t)$ is a major function to $f\{x(t)\}x'(t)$. If $\phi(x)$ is a minor function to $f(x)$ it can be shown in a similar way that $\Theta(t)=\phi\{x(t)\}$ is a minor function to $f\{x(t)\}x'(t)$. Furthermore, since the common bound of the functions $\psi(x)$ and $\phi(x)$ is $F(x)$, the common bound of the functions $\chi(t)$ and $\Theta(t)$ is $F\{x(t)\}$. Thus the function $f\{x(t)\}x'(t)$ is Perron integrable to $F\{x(t)\}$ and

$$F(b) - F(a) = \int_a^b f(x)dx = \int_\alpha^\beta \{x(t)\}x'(t)dt.$$

ACADIA UNIVERSITY

ON PERRON INTEGRATION

E. J. McSHANE

The definition of integral here presented had its origin in an unsuccessful attempt to establish the theorem on integration by parts[1] (Theorem 4.1 of this note) for the Perron integral, without detouring through the special Denjoy integral. In order to avoid the difficulties which I could not overcome, I amended the definition of the integral; the resulting definition I presented at the Oslo congress in 1936. Recently I have found a proof that the "new" integral is actually equivalent to that of Perron.

As compared with Perron's definition, the new definition has the slight disadvantage that it requires four associated functions instead of two; in all other respects the proofs of theorems for the Perron integral carry over unaltered. It has the advantage that it permits us to prove the general theorem on integration by parts (Theorem 4.1) without recourse to the deep-lying equivalence with the special Denjoy integral. This theorem is important, not only in itself, but because it also contains the corollary that if $f(x)$ is Perron integrable and $g(x)$ of limited total variation, their product is Perron integrable. Also it leads at once to the second theorem of the mean for Perron integrals.

1. **Definition of the integral.** Let $f(x)$ be a function defined on an interval $[a, b]$ and assuming values which are real numbers or $+\infty$ or $-\infty$. A set of four functions $\phi^i(x)$ $(i=1, 2, 3, 4)$ is a *tetrad adjoined to* $f(x)$ *on* $[a, b]$ if the following conditions are satisfied.

(1.1a) *The* $\phi^i(x)$ *are continuous on* $[a, b]$ *and all vanish at* $x=a$.

(1.1b) *For all except at most a denumerable collection of values of* x *the relations*

$$-\infty \neq D_+\phi^1, \qquad -\infty \neq D_-\phi^2,$$
$$+\infty \neq D^-\phi^3, \qquad +\infty \neq D^+\phi^4$$

are valid.

(1.1c) *For all except at most a denumerable collection of values of* x *the relations*

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[1] For a proof of the theorem with the added hypothesis that $g'(x)$ is finite except for a denumerable set, see R. L. Jeffery, *Perron integrals*, this Bulletin, vol. 48 (1942). pp. 714–717.

$$D_+\phi^1(x) \geqq f(x), \qquad D_-\phi^2(x) \geqq f(x),$$
$$D^-\phi^3(x) \leqq f(x), \qquad D^+\phi^4(x) \leqq f(x)$$

are valid.

In case the functions $\phi^i(x)$ also satisfy the conditions

$$|\phi^i(b) - \phi^j(b)| < \epsilon, \qquad i, j = 1, 2, 3, 4,$$

the tetrad will be said to be ϵ-adjoined to f. Functions satisfying the conditions imposed on $\phi^1, \phi^2, \phi^3, \phi^4$, respectively, will be called right majors, left majors, left minors, right minors, respectively.

(1.2) *The function $f(x)$ is P^*-integrable on $[a, b]$ if for every positive ϵ there is a tetrad ϵ-adjoined to f on $[a, b]$.*

Before defining the integral we establish an essential lemma.

LEMMA 1.1. *If ψ is a major function (right or left) for f, and ϕ is a minor function (right or left) for f, then $\psi - \phi$ is monotonic non-decreasing.*

Suppose first that ψ is a right major and ϕ a right minor. By (1.1b) and (1.1c), except on a denumerable set we have

$$D_+(\psi - \phi) \geqq D_+\psi - D^+\phi \geqq 0.$$

So[2] $\psi - \phi$ is non-decreasing. A similar proof applies if ψ is a left major and ϕ a left minor.

Suppose next that ψ is a right major and ϕ a left minor. By a theorem of G. C. Young,[3] the inequality $D_+\phi \leqq D^-\phi$ holds except at most on a denumerable set. So by (1.1b) and (1.1c) we have

$$D^+(\psi - \phi) \geqq D_+\psi - D_+\phi \geqq 0$$

except on a denumerable set, and[2] $\psi - \phi$ is non-decreasing.

A similar proof applies if ψ is a left major and ϕ a right minor.

COROLLARY 1.2. *If $f(x)$ is P^*-integrable on $[a, b]$, it is also P^*-integrable on $[a, x]$ for all x such that $a < x \leqq b$.*

For by Lemma 1.1 every tetrad which is ϵ-adjoined to f on $[a, b]$ is also ϵ-adjoined to f on $[a, x]$.

LEMMA 1.3. *If $f(x)$ is P^*-integrable on $[a, b]$, there is a function $F(x)$ on $[a, b]$ which is the* g.l.b. *of all right majors, the* g.l.b. *of all left majors, the* l.u.b. *of all left minors and the* l.u.b. *of all right minors. The function $F(x)$ is continuous.*

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[2] Saks, *Theory of the Integral*, p. 204.

[3] Hobson, *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 392.

Let ϕ be any minor of f. If ϕ^1 is a right major, for every x in $[a, b]$ we have

$$\phi^1(x) - \phi(x) \geqq \phi^1(a) - \phi(a) = 0;$$

so $\phi(x)$ is a lower bound for right majors. If we define $F(x)$ to be the g.l.b. of all right majors, then $F(x) \geqq \phi(x)$. Here ϕ is any minor, so F is an upper bound for all minors. If $\epsilon > 0$, there is a tetrad ϕ^i which is ϵ-adjoined to f, so

$$(1.4) \qquad F(x) - \phi^4(x) \leqq \phi^1(x) - \phi^4(x) \leqq \phi^1(b) - \phi^4(b) < \epsilon,$$

and likewise $F(x) - \phi^3(x) < \epsilon$. That is, $F(x)$ is the l.u.b. of all right minors and also is the l.u.b. of all left minors. Interchanging majors and minors shows that F is also the g.l.b. of all left majors.

If we let ϵ approach 0 through a sequence of positive values, by (1.4) the corresponding right minors $\phi^4(x)$ converge uniformly to $F(x)$, which therefore is continuous.

DEFINITION 1.4. *If $f(x)$ is P^*-integrable on $[a, b]$, and $F(x)$ is the function defined in Lemma* 1.2, *then the P^*-integral of f over $[a, b]$ is defined to be*

$$(P^*) \int_a^b f(x)dx = F(b).$$

Henceforth we omit the P^* before the integral.

COROLLARY 1.5. *With the hypotheses of Definition* 1.4, *if $a < x \leqq b$ then*

$$\int_a^x f(x)dx = F(x).$$

2. **Properties of the integral.** We can now readily establish the elementary properties of the P^*-integral; if f_1 and f_2 are P^*-integrable, so is kf_1 and so is $f_1 \pm f_2$ if defined (that is, if never of the form $\infty - \infty$), and the integrals have the values to be expected; f_1 is P^*-integrable over every subinterval of $[a, b]$, and is an additive function of intervals; and the function $f_0(x) = f_1(-x)$ is P^*-integrable over $[-b, -a]$, and its integral is equal to

$$\int_a^b f_1(x)dx.$$

The proofs differ from the corresponding proofs for the Perron integral only in minor details.

Furthermore, we can readily show that every P^*-integrable function is finite almost everywhere, is measurable, and is almost everywhere equal to the derivative of its indefinite P^*-integral. The proof in Sak's *Theory of the Integral*, p. 202, needs only obvious changes.

An alternative form of definition is given in the next theorem.

THEOREM 2.1. *Let $f(x)$ be defined on an interval $[a, b]$. In order that $f(x)$ be P^*-integrable on $[a, b]$ it is necessary and sufficient that for every positive number ϵ there exist four functions $\psi^i(x)$ $(i=1, 2, 3, 4)$ which satisfy* (1.1a) *and* (1.1b), *satisfy the inequalities in* (1.1c) *for almost all x, and are such that*

$$|\psi^i(b) - \psi^j(b)| < \epsilon, \qquad i, j = 1, 2, 3, 4. \tag{2.1}$$

In this case, the P^*-integral of f over $[a, x]$ is the common g.l.b. *of all functions $\psi^1(x)$ and $\psi^2(x)$ and the common* l.u.b. *of all functions $\psi^3(x)$ and $\psi^4(x)$ which satisfy the conditions stated.*

The proof is essentially that given by Bauer[4] for the Perron integral. As an obvious corollary, if $f(x)$ is P^*-integrable on $[a, b]$ so is every function equivalent to f, and the integrals are the same.

3. **Substitution.** Next we establish the following theorem on change of independent variable.

THEOREM 3.1. *Let $f(x)$ be P^*-integrable on an interval $[a, b]$. Let $g(y)$ be a function defined on an interval $[\alpha, \beta]$ and possessing the following properties.*

(a) *$g(y)$ is continuous and non-decreasing on $[\alpha, \beta]$.*

(b) *$g(\alpha)=a$, $g(\beta)=b$.*

(c) *$\overline{D}g(y) < \infty$ except at most on a denumerable subset of $[\alpha, \beta]$.*

Then $f(g(y))\, g(y)$ is P^*-integrable over $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(y))g(y)dy.$$

For each positive ϵ there is a tetrad $[\phi^i]$ which is ϵ-adjoined to $f(x)$ on $[a, b]$. Define

$$\psi^i(y) = \phi^i(g(y)), \qquad i = 1, \cdots, 4; \alpha \leqq y \leqq \beta.$$

It is obvious that the ψ^i are continuous on $[a, b]$ and vanish at $y=\alpha$. Let E_1 consist of all y in $[\alpha, \beta]$ such that there exists a point y' with $y<y' \leqq \beta$ and $g(y')=g(y)$. The rest of the set $[\alpha, \beta]$ we

[4] H. Bauer, *Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen*, Monatshefte für Mathematik und Physik, vol. 26 (1915), pp. 153–198; also Saks, op. cit., p. 203.

denote by E_2; if y is in E_2 then $g(y')>g(y)$ for all y' such that $y<y'\leqq\beta$.

If y is in E_1, the function $g(y)$ remains constant on an interval $[y, y']$, so each $\psi^i(y)$ also is constant on $[y, y']$. Therefore

$$-\infty < D_+\psi^1(y) = 0 = f(g(y))\dot{g}(y), \tag{3.1}$$

$$+\infty > D^+\psi^4(y) = 0 = f(g(y))\dot{g}(y). \tag{3.2}$$

If y is in E_2, for each y' such that $y<y'\leqq\beta$ we have

$$\frac{\psi^1(y')-\psi^1(y)}{y'-y} = \frac{\phi^1(g(y'))-\phi^1(g(y))}{g(y')-g(y)}\cdot\frac{g(y')-g(y)}{y'-y}. \tag{3.3}$$

By definition (1.1), $D_+\phi^1>-\infty$ except at most on a denumerable set X_0. If x is in X_0, the equation $g(y)=x$ can have at most one solution in E_2, for of two solutions lying in $[\alpha, \beta]$ the lesser is in E_1 by definition. So the set of such solutions is denumerable. To it we add the at most denumerable set on which $\overline{D}g=\infty$; the sum Y_0 is at most denumerable. If y is in E_2-Y_0, we let $y'\to y^+$, and from (3.3) obtain

$$D_+\psi^1 \neq -\infty. \tag{3.4}$$

Moreover, if $g'(y)$ exists, as it does almost everywhere in $[\alpha, \beta]$, the same limiting process yields

$$D_+\psi^1(y) = D_+\phi^1(g(y))\dot{g}(y). \tag{3.5}$$

Combining this with (3.1) and (1.1), we see that $D_+\psi^1(y)>-\infty$ except at most on a denumerable set and

$$D_+\psi^1(y) \geqq f(g(y))\dot{g}(y)$$

on almost all of $[\alpha, \beta]$.

The other functions ψ^i can be discussed analogously; we thus see that they satisfy the first three conditions of Theorem 2.1. The last condition (2.1) is trivial, since

$$\psi^i(\beta) - \psi^j(\beta) = \left|\phi^i(b) - \phi^j(b)\right| < \epsilon.$$

So by Theorem 2.1 the product $f(g(y))\dot{g}(y)$ is P^*-integrable on $[\alpha, \beta]$. Moreover, its integral over $[\alpha, \beta]$ lies between $\psi^4(\beta)$ and $\psi^1(\beta)$, that is, between $\phi^4(b)$ and $\phi^1(b)$. So does the integral of $f(x)$ over $[a, b]$. Since $\phi^4(b)$ and $\phi^1(b)$ differ by less than an arbitrary ϵ, the two integrals are equal.

Theorem 3.1 has an obvious analogue for monotonic non-increasing functions $g(y)$. Also, it is worth observing that the hypotheses

on $g(y)$ force it to be absolutely continuous. For if we set $f \equiv 1$ in Theorem 3.1 we find that

$$g(\beta) - g(\alpha) = a - b = \int_a^b g(y)dy.$$

Since g is non-negative, the integral on the right reduces to a Lebesgue integral, as we can show by the usual proof for the Perron integral. This equation implies the absolute continuity of $g(y)$.

4. **Integration by parts.** The principal reason for devising Definition 1.1 was to enable us to prove the following theorem.

THEOREM 4.1. *Let $f(x)$ be P^*-integrable on $[a, b]$, and let $F(x)$ be its indefinite P^*-integral. If $g(x)$ is of limited total variation on $[a, b]$, then the product $f(x)g(x)$ is P^*-integrable, and*

$$\int_a^b f(x)g(x)dx = F(b)g(b) - \int_a^b F(x)dg(x), \tag{4.1}$$

the last integral being a Stieltjes integral.

The function $g(x)$ can be written in either of the forms

$$g(x) = g_1(x) - g_2(x),$$
$$g(x) = g_3(x) - g_4(x),$$

where the g_i are positive, g_1 and g_2 are monotonic non-increasing and g_3 and g_4 are monotonic non-decreasing. Let ϕ^i be a tetrad ϵ-adjoined to f on $[a, b]$. We define sixteen functions by the formulas

$$\psi_j^i(x) = \phi^i(x)g_j(x) - \int_a^x \phi^i(x)dg_j(x), \qquad i, j = 1, 2, 3, 4. \tag{4.2}$$

It is evident that these all vanish at $x=a$. If x and x' are in $[a, b]$, we can apply the first mean-value theorem to the integral in (4.2) to obtain

$$\begin{aligned} \psi_j^i(x') - \psi_j^i(x) &= \phi^i(x')g_j(x') - \phi^i(x)g_j(x) - \phi^i(\bar{x})[g_j(x') - g_j(x)] \\ &= g_j(x')[\phi^i(x') - \phi^i(x)] \\ &\quad - [g_j(x') - g_j(x)](\phi^i(\bar{x}) - \phi^i(x)), \end{aligned} \tag{4.3}$$

where $\bar{x}$ is between x and x' (inclusive). Since ϕ^i is continuous and g_j is bounded, this shows that $\psi_j^i(x)$ is continuous.

Suppose that $g'(x)$ is defined and finite; this is true for almost all x. We divide both numbers of (4.3) by $x'-x$, writing the $x'-x$

under the square-bracketed terms on the right. The second term on the right tends to zero, so

$$D^+\psi_j^i(x) = g_j(x)D^+\phi^i(x), \tag{4.4}$$

and likewise for the three other Dini derivates.

If $\bar{x} \neq x$ equation (4.3) yields

$$\frac{\psi_j^i(x') - \psi_j^i(x)}{x' - x} = g_j(x')\frac{\phi^i(x') - \phi^i(x)}{x' - x} + (g_j(x) - g_j(x'))\cdot\frac{\phi^i(\bar{x}) - \phi^i(x)}{\bar{x} - x}\cdot\frac{\bar{x} - x}{x' - x}. \tag{4.5}$$

This is still valid if $\bar{x} = x$, provided that we assign a value, say 0, to the middle factor in the last term. Henceforth we assume that x is in the set on which the inequalities in (1.1b) hold; this rejects at most a denumerable set of x. Let x' approach x from the right. The last factor in (4.5) is non-negative and bounded; so are the factors involving g_i if $j = 1$ or 2. If $i = 1$ the fractions involving ϕ^i are bounded below, since $D_+\phi^i(x) > -\infty$; if $i = 4$ they are bounded above. Hence the left member is bounded below if $i = 1$, above if $i = 4$, and

$$D_+\psi_j^1 > -\infty, \qquad D^+\psi_j^4 < \infty, \qquad j = 1, 2. \tag{4.6}$$

In a similar way, if $x' < x$ and $j = 3$ or 4 we can establish

$$D_-\psi_j^2 > -\infty, \qquad D^-\psi_j^3 < \infty, \qquad j = 3, 4. \tag{4.7}$$

Now we define

$$\begin{aligned} \psi^1 &= \psi_1^1 - \psi_2^4, \\ \psi^2 &= \psi_2^2 - \psi_4^3, \\ \psi^3 &= \psi_3^3 - \psi_4^2, \\ \psi^4 &= \psi_1^4 - \psi_2^1. \end{aligned} \tag{4.8}$$

From (4.6) and (4.7) we find that these satisfy (1.1b). From (4.4), together with (1.1b), we find that the inequalities of (1.1c) hold for almost all x, and (1.1a) has already been established.

Since F is the g.l.b. of majors and the l.u.b. of minors, we find

$$\psi_j^4(b) \leqq F(b)g_j(b) - \int_a^b F(x)dg_j(x) \leqq \psi_j^1(b), \qquad j = 1, 2.$$

Hence by (4.8)

$$\psi^1(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^4(b). \tag{4.9}$$

Likewise

$$\psi^2(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^3(b). \tag{4.10}$$

But

$$|\psi_j^1(b) - \psi_j^4(b)| = |[\phi^1(b) - \phi^4(b)]g_j(b) - \int_a^b [\phi^1(x) - \phi^4(x)]dg_j(x)|$$

$$\leqq \epsilon[|g_j(b)| + |g_j(b) - g_j(a)|],$$

and, denoting the coefficient of ϵ by K_j, this implies

$$|\psi^1(b) - \psi^4(b)| < \epsilon(K_1 + K_2).$$

Likewise

$$|\psi^2(b) - \psi^3(b)| < \epsilon(K_3 + K_4).$$

Since ϵ is arbitrary, this implies that with (4.9) and (4.10) all four numbers $\psi^i(b)$ lie arbitrarily close to

$$F(b)g(b) - \int_a^b F(x)dg(x).$$

This establishes the inequality (2.1) of Theorem 2.1, hence proves the P^*-integrability of $f(x)\ g(x)$, and also establishes equation (4.1).

5. **Equivalence of the P^*-integral and the Perron integral.** It is evident that every function which is[5] Perron integrable on an interval $[a, b]$ is also P^*-integrable, and the integrals are equal. For the Perron major functions serve simultaneously as right and left majors, and the Perron minors serve as right and left minors.

The converse is less evident. It requires slight generalizations of two known theorems.

LEMMA 5.1. *If $F(x)$ is a function which at all but at most a denumerable subset of a set M satisfies one of the inequalities*

$$D^+F < \infty, \qquad D^-F < \infty,$$

$$D_+F > -\infty, \qquad D_-F > -\infty,$$

then $F(x)$ is VBG_* on M.

The proof is essentially that given on p. 235 of Saks' *Theory of the Integral*. His equation (10.3) is replaced by

[5] By the Perron integral we mean the integral originally defined by Perron and studied by Bauer (loc. cit.); this is called the $\mathcal{P}_0$ integral by Saks, and is equivalent to the $\mathcal{P}$ integral (Saks, op. cit., p. 252).

$$0 < t - x < 1/n \text{ implies } [F(t) - F(x)]/(t - x) \leqq n;$$

the remainder of the proof is unaltered. The next lemma is a slight generalization of a theorem of Marcinkiewicz.

LEMMA 5.2. *If $f(x)$ is measurable on $[a, b]$, and has either a left major or a right major, and also has either a left minor or a right minor, then $f(x)$ is Perron integrable on $[a, b]$.*

The proof is that given by Saks, op. cit., p. 253; the principal change is that the reference to his Theorem 10.1 is replaced by a reference to our Lemma 5.1.

Since every P^*-integrable function $f(x)$ is measurable and has right majors and right minors, it is also Perron integrable by Lemma 5.2, and the equivalence of the integrals is established.

UNIVERSITY OF VIRGINIA

ON THE LEAST PRIMITIVE ROOT OF A PRIME

LOO-KENG HUA

It was proved by Vinogradow[1] that the least positive primitive root $g(p)$ of a prime p is $O(2^m p^{1/2} \log p)$ where m denotes the number of different prime factors of $p-1$. In 1930 he[2] improved the previous result to

$$g(p) = O(2^m p^{1/2} \log \log p),$$

or more precisely,

$$g(p) \leqq 2^m \frac{p-1}{\phi(p-1)} p^{1/2}.$$

It is the purpose of this note, by introducing the notion of the average of character sums,[3] to prove that if $h(p)$ denotes the primitive root with the least absolute value, mod p, then

$$|h(p)| < 2^m p^{1/2};$$

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[1] See, Landau, *Vorlesungen über Zahlentheorie*, vol. 2, part 7, chap. 14. The original papers of Vinogradow are not available in China.

[2] Comptes Rendus de l'Académie des Sciences de l'URSS, 1930, pp. 7–11.

[3] The present note may be regarded as an introduction of a method which has numerous applications.

and that for $p \equiv 1 \pmod 4$, we have

$$g(p) < 2^m p^{1/2},$$

while, for $p \equiv 3 \pmod 4$, we have

$$g(p) < 2^{m+1} p^{1/2}.$$

Since

$$\frac{p-1}{\phi(p-1)} \geqq 2,$$

the result is always better than that due to Vinogradow.

LEMMA 1. *Let $p>2$, $1 \leqq A < p$. For each non-principal character*[4] $\chi(n)$, mod p, *we have*

$$\frac{1}{A+1} \left| \sum_{a=0}^{A} \sum_{n=-a}^{a} \chi(n) \right| \leqq p^{1/2} - \frac{A+1}{p^{1/2}}.$$

PROOF. Let $\epsilon = e^{2\pi i/p}$ and let

$$\tau(\chi) = \sum_{h=1}^{p-1} \chi(h)\epsilon^h.$$

It is known that

$$|\tau(\chi)| = p^{1/2}.$$

For $p \nmid n$, we have

$$\sum_{h=1}^{p-1} \bar{\chi}(h)\epsilon^{hn} = \chi(n) \sum_{h=1}^{p-1} \bar{\chi}(hn)\epsilon^{hn} = \chi(n) \sum_{h=1}^{p-1} \bar{\chi}(h)\epsilon^h = \chi(n)\tau(\bar{\chi}).$$

The formula holds also for $p \mid n$, since $\chi(n) = 0$ for $p \mid n$ and

$$\sum_{h=1}^{p-1} \bar{\chi}(h) = 0.$$

Thus

$$\tau(\bar{\chi}) \sum_{a=0}^{A} \sum_{n=-a}^{a} \chi(n) = \sum_{h=1}^{p-1} \bar{\chi}(h) \sum_{a=0}^{A} \sum_{n=-a}^{a} \epsilon^{hn} = \sum_{h=1}^{p-1} \bar{\chi}(h) \left(\frac{\sin (A+1)\pi h/p}{\sin \pi h/p} \right)^2.$$

[4] See, for example, Landau loc. cit., vol. 1, pp. 83–87.

Consequently

$$p^{1/2}\left|\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n)\right| \leqq \sum_{h=1}^{p-1}\left(\frac{\sin(A+1)\pi h/p}{\sin \pi h/p}\right)^2$$
$$= \sum_{h=1}^{p-1}\sum_{a=0}^{A}\sum_{n=-a}^{a}\epsilon^{hn}$$
$$= \sum_{a=0}^{A}\sum_{n=-a}^{a}\left(\sum_{h=1}^{p}\epsilon^{hn}-1\right)$$
$$= (A+1)p-(A+1)^2.$$

LEMMA 2. *Let $p>2$, $1\leqq A<(p-1)/2$. Then, for each non-principal character*, mod p, *we have*

$$\frac{1}{A+1}\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi(n)\right| \leqq p^{1/2}-\frac{A+1}{p^{1/2}}.$$

PROOF. As in Lemma 1, we have

$$p^{1/2}\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi(n)\right| = \left|\sum_{h=1}^{p-1}\bar{\chi}(h)e^{2\pi i h(A+1)/p}\left(\frac{\sin(A+1)\pi h/p}{\sin \pi h/p}\right)^2\right|$$
$$\leqq \sum_{h=1}^{p-1}\left(\frac{\sin(A+1)\pi h/p}{\sin \pi h/p}\right)^2$$
$$= (A+1)p-(A+1)^2.$$

LEMMA 3. *Let $p>2$. If n is not a primitive root*, mod p, *then*

$$\sum_{k\mid p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\chi^{(k)}(n)=0,$$

where $\chi^{(k)}$ runs over all characters χ satisfying the condition that k is the least positive integer such that $(\chi)^k$ is the principal character.

(See Landau, loc. cit., p. 496. The condition $1\leqq n<p$ there mentioned is not necessary.)

THEOREM 1. *We have $|h(p)|<2^m p^{1/2}$.*

PROOF. Let $p>2$. By Lemma 3, we have

$$0=\sum_{k\mid p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\sum_{a=0}^{|h(p)|-1}\sum_{n=-a}^{a}\chi^{(k)}(n).$$

For $k=1$, the right-hand side gives

$$\sum_{a=0}^{|h(p)|-1} \sum_{n=-a}^{a} \chi^{(1)}(n) = \sum_{a=0}^{|h(p)|-1} 2a.$$
$$= |h(p)|^2 - |h(p)|.$$

On the other hand, for $k \neq 1$, we have, by Lemma 1 with $A = |h(p)| - 1$,

$$\left| \sum_{a=0}^{|h(p)|-1} \sum_{n=-a}^{a} \chi^{(k)}(n) \right| \leqq |h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}}.$$

Therefore

$$|h(p)|^2 - |h(p)| \leqq \left(|h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}} \right) \sum_{k|p-1} \frac{|\mu(k)|}{\phi(k)} \phi(k)$$
$$= 2^m \left(|h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}} \right).$$

Then

$$|h(p)| \leqq \frac{2^m p^{1/2} + 1}{1 + 2^m/p^{1/2}} < 2^m p^{1/2}.$$

COROLLARY. *For $p \equiv 1 \pmod 4$, we have $g(p) = |h(p)| < 2^m p^{1/2}$.*

PROOF. We have to show that $|h(p)|$ is a primitive root. Suppose it is not. Then $-|h(p)|$ is a primitive root and $|h(p)|$ belongs to an exponent l where $l | (p-1)$ and $l < p-1$, that is,

$$|h(p)|^l \equiv 1 \pmod p,$$
$$(h(p))^{2l} \equiv 1 \pmod p.$$

Thus $2l = p-1$ and $|h(p)|^{(p-1)/2} \equiv 1 \pmod p$ so that $|h(p)|$ is a quadratic residue. Since -1 is a quadratic residue, mod p, $-|h(p)|$ is also a quadratic residue and $\{-|h(p)|\}^{(p-1)/2} \equiv 1 \pmod p$. This contradicts the fact that $-|h(p)|$ is a primitive root.

REMARK. Sometimes Theorem 1 may be improved by the fact that

$$\sum_{n=-a}^{a} \chi^{(k)}(n) = 0,$$

for $\chi^{(k)}(-1) = -1$ and hence $\chi^{(k)}(n) = -\chi^{(k)}(-n)$. Thus for $p \equiv 3 \pmod 4$,

$$|h(p)| < 2^{m-1} p^{1/2}.$$

In fact, we have $g^{(p-1)/2} \equiv -1 \pmod p$ and $\chi^{(k)}(g) = e^{2\pi i \Omega/k}$. Since

$$-1 = \chi^{(k)}(g^{(p-1)/2}) = e^{\pi i (p-1)\lambda/k},$$

we have $2\nmid(p-1)\lambda/k$. The terms appearing in the formula of Lemma 3 are those with square-free k. Thus $\chi^{(k)}(-1)=-1$ holds only for the case $p\equiv 3 \pmod 4$, and $2\nmid\lambda$. Thus

$$\sum_{a=0}^{|h(p)|-1}\sum_{n=-a}^{a}\chi^{(k)}(n)=0 \qquad \text{for } 2\mid k.$$

Therefore

$$|h(p)|^2-|h(p)| \leqq \left(|h(p)|p^{1/2}-\frac{|h(p)|^2}{p^{1/2}}\right)\sum_{k\mid(p-1)/2}|\mu(k)|$$

$$=2^{m-1}\left(|h(p)|p^{1/2}-\frac{|h(p)|^2}{p^{1/2}}\right).$$

Then

$$|h(p)| \leqq \frac{2^{m-1}p^{1/2}+1}{1+2^{m-1}/p^{1/2}} < 2^{m-1}p^{1/2}.$$

THEOREM 2. *We have* $g(p)<2^{m+1}p^{1/2}$.

PROOF. Let A be the greatest integer not exceeding $(g-1)/2$. Then

$$0=\sum_{k\mid p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(k)}(n).$$

For $k=1$, the right-hand side gives

$$\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(1)}(n)=\sum_{a=0}^{A}(2a+1)=(A+1)^2.$$

For $k\neq 1$, we have

$$\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(k)}(n)\right| \leqq (A+1)p^{1/2}-\frac{1}{p^{1/2}}(A+1)^2.$$

Therefore, as in the proof of Theorem 1, we have

$$(A+1)^2 < 2^m\left((A+1)p^{1/2}-\frac{1}{p^{1/2}}(A+1)^2\right),$$

$$(g-1)/2 < A+1 \leqq \frac{2^m p^{1/2}}{1+2^m/p^{1/2}},$$

that is,

$$g \leqq \frac{2^{m-1}p^{1/2}}{1+2^m/p^{1/2}}+1 < 2^{m+1}p^{1/2}.$$

NATIONAL TSING HUA UNIVERSITY

ON THE LEAST SOLUTION OF PELL'S EQUATION

LOO-KENG HUA

Let x_0, y_0 be the least positive solution of Pell's equation

$$x^2 - dy^2 = 4,$$

where d is a positive integer, not a square, congruent to 0 or 1 (mod 4). Let $\epsilon = (x_0 + d^{1/2}y_0)/2$. It was proved by Schur[1] that

$$\epsilon < d^{d^{1/2}}, \tag{1}$$

or, more precisely,

$$\log \epsilon < d^{1/2}((1/2)\log d + (1/2)\log\log d + 1)\,. \tag{2}$$

He deduced (1) from (2) by the property that

$$d^{1/2}((1/2)\log d + (1/2)\log\log d + 1) < d^{1/2}\log d$$

for $d > 244.69\cdots$, and, for $d \leqq 244$, (1) is established by direct computation. It is the object of the present note to establish a slightly better result that

$$\log \epsilon < d^{1/2}((1/2)\log d + 1). \tag{3}$$

Thus (1) follows immediately without any calculation. The method used is that described in the preceding paper.

Let $(d\,|\,r)$ be Kronecker's symbol. (We extend the definition to include negative values of r by the relation $(d\,|\,r_1) = (d\,|\,r_2)$ for $r_1 \equiv r_2 \pmod{d}$.)

Let f denote the fundamental discriminant related to d, that is,

$$d = m^2 f,$$

where f is not divisible by a square of odd prime and is either odd, or congruent to 8 or congruent to 12 (mod 16).

LEMMA 1. *For $d > 0$, we have*

$$\left(\frac{d}{r}\right) = \left(\frac{d}{-r}\right).$$

PROOF. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Theorem 101.

LEMMA 2. *We have*

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[1] Göttingen Nachrichter, 1918, pp. 30–36.

$$\sum_r \left(\frac{f}{r}\right) e^{2\pi i n r/f} = \left(\frac{f}{n}\right) f^{1/2},$$

where r runs over a complete residue system, mod f.

PROOF. Landau, loc. cit., Theorem 215.

LEMMA 3. *We have*

$$\frac{1}{A^*+1}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| \leqq \frac{1}{2}\left(f^{1/2} - \frac{A^*+1}{f^{1/2}}\right),$$

where A^* is the least positive residue of A, mod f.

PROOF. (See Lemma 1 of the preceding paper.) We have, by Lemma 2,

$$\begin{aligned} f^{1/2}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right) &= \frac{1}{2}f^{1/2}\sum_{a=0}^{A}\sum_{n=-a}^{a}\left(\frac{f}{n}\right) \\ &= \frac{1}{2}\sum_{a=0}^{A}\sum_{n=-a}^{a}\sum_{r=1}^{f}\left(\frac{f}{r}\right)e^{2\pi i n r/f} \\ &= \frac{1}{2}\sum_{r=1}^{f}\left(\frac{f}{r}\right)\sum_{a=0}^{A}\sum_{n=-a}^{a}e^{2\pi i n r/f}. \end{aligned}$$

Then

$$\begin{aligned} f^{1/2}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r=1}^{f-1}\left|\sum_{a=0}^{A}\sum_{n=-a}^{a}e^{2\pi i n r/f}\right| \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin(A+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin(A^*+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\sum_{a=0}^{A_0}\sum_{n=-a}^{a}e^{2\pi i n r/f} \\ &= \frac{1}{2}((A^*+1)f - (A^*+1)^2), \end{aligned}$$

since

$$\sum_{r=1}^{f-1}e^{2\pi i n r/f} = \sum_{r=1}^{f}e^{2\pi i n r/f} - 1 = \begin{cases} -1 & \text{if } f\nmid n, \\ f-1 & \text{if } f\mid n. \end{cases}$$

LEMMA 4. *For any discriminant $d>0$ and $A>d^{1/2}$, we have*

$$\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| \leqq \frac{1}{2}Ad^{1/2}.$$

PROOF. It is well known that[2]

$$\left(\frac{d}{n}\right) = \left(\frac{f}{n}\right)\sum_{r|(m,n)}\mu(r).$$

Then

$$\begin{aligned}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right) &= \sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\sum_{r|(m,n)}\mu(r)\\ &= \sum_{r|m}\mu(r)\sum_{a=1}^{A}\sum_{n=1,\,r|n}^{a}\left(\frac{f}{n}\right) = \sum_{r|m}\mu(r)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{rn}\right)\\ &= \sum_{r|m}\mu(r)\left(\frac{f}{r}\right)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right).\end{aligned}$$

Then, by Lemma 2,

$$\begin{aligned}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r|m}\left|\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r|m}r\left|\sum_{b=1}^{[A/r]}\sum_{n=1}^{b}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r|m}r\left(\left(\left[\frac{A}{r}\right]+1\right)f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^2\right)\\ &\leqq \frac{1}{2}\sum_{r|m}r\cdot\frac{A}{r}f^{1/2} \leqq \frac{1}{2}Af^{1/2}m = \frac{1}{2}Ad^{1/2},\end{aligned}$$

since we have $f^{1/2}r < f^{1/2}m < A$,

$$f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^2 < f^{1/2} - \frac{1}{f^{1/2}}\cdot f = 0$$

and

$$\sum_{r|m}1 \leqq m.$$

LEMMA 5. *We have*

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n} < \frac{1}{2}\log d + 1.$$

[2] This follows from the fact that $\sum_{d|a}\mu(d) = 0$ or 1 according as $a > 1$ or $a = 1$.

PROOF. For $n \geqq 1$ let

$$S(n) = \sum_{a=1}^{n} \sum_{m=1}^{a} \left(\frac{d}{m}\right),$$

and let $S(0) = S(-1) = 0$. Then we have

$$S(n) - 2S(n-1) + S(n-2) = \left(\frac{d}{n}\right), \qquad n \geqq 1,$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n} &= \sum_{n=1}^{\infty}\{S(n) - 2S(n-1) + S(n-2)\}\frac{1}{n} \\ &= \sum_{n=1}^{\infty} S(n)\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\ &= \sum_{n=1}^{\infty} \frac{2S(n)}{n(n+1)(n+2)}.\end{aligned}$$

We divide the series into two parts

$$S_1 = \sum_{n=1}^{A-1}, \qquad S_2 = \sum_{n=A}^{\infty}.$$

Since

$$|S(n)| \leqq \sum_{a=1}^{n} \sum_{m=1}^{a} 1 = \frac{n(n+1)}{2},$$

it follows that

$$|S_1| \leqq \sum_{n=1}^{A-1} \frac{1}{n+2}.$$

If $A > d^{1/2}$ we have by Lemma 4

$$|S_2| < \sum_{n=A}^{\infty} \frac{nd^{1/2}}{n(n+1)(n+2)} = \frac{d^{1/2}}{A+1}.$$

Hence

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\right| &\leqq \sum_{n=1}^{A-1} \frac{1}{n+2} + \frac{d^{1/2}}{A+1} \\ &= \sum_{m=1}^{A-1} \frac{1}{m} - 1 - \frac{1}{2} + \frac{1}{A} + \frac{1}{A+1} + \frac{d^{1/2}}{A+1} \\ &\leqq \log(A-1) - \frac{1}{2} + \frac{1}{A} + \frac{d^{1/2}+1}{A+1}.\end{aligned}$$

Taking $A=[d^{1/2}]+1$ we have

$$\left|\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\right| \leqq \log d^{1/2} - \frac{1}{2} + \frac{1}{d^{1/2}} + \frac{d^{1/2}+1}{d^{1/2}+1}$$

$$= \frac{1}{2}\log d + \frac{1}{2} + \frac{1}{d^{1/2}} < \frac{1}{2}\log d + 1$$

since $d \geqq 5$.

THEOREM 1. *We have*

$$\log \epsilon < d^{1/2}((1/2)\log d + 1).$$

PROOF. It is known that the number $h(d)$ of classes of non-equivalent quadratic forms with determinant $d>0$, is given by

$$h(d) = \frac{d^{1/2}}{\log \epsilon}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}.$$

Since $h(d) \geqq 1$, we have the theorem.

THEOREM 2 (Schur). *We have*

$$\log \epsilon \leqq d^{1/2}\log d.$$

PROOF. For $d>e^2$, the theorem follows from Theorem 1. If $d<e^2$, then $d=5$. Evidently $\epsilon=(3+5^{1/2})/2$ and

$$\log \epsilon < 5^{1/2}\log 5.$$

NATIONAL TSING HUA UNIVERSITY

A NOTE ON THE DIOPHANTINE PROBLEM OF FINDING FOUR BIQUADRATES WHOSE SUM IS A BIQUADRATE

J. O. PATTERSON

The problem is to assign integers satisfying the relation

$$x^4 = y^4 + a^4 + b^4 + c^4. \tag{1}$$

Granted that the five integers have no common divisor, it follows that x and one quantity on the right (here taken as y) are not multiples of 5; and it can be shown that (1) without loss of generality can be replaced by

$$x^4 = y^4 + 5^4t^4(\alpha^4 + \beta^4 + \gamma^4). \tag{2}$$

In this expression x is an odd number prime to y; y may be odd or even; and neither is divisible by 5, as stated above.

Let $(x^4-y^4)/5^4t^4=d$; which for values satisfying (2) will be an integer.

Evidently x and y must satisfy the congruence $x^4\equiv y^4 \pmod{5^4}$. Hence, for values of x up to any required magnitude, there are corresponding values of y;[1] and the resultant values of d can be found and tabulated.

If (2) is satisfied, then $d=\alpha^4+\beta^4+\gamma^4$; and from the elementary properties of the sum of three biquadrates it is seen that many values of d may be rejected at once: for example, all those having for the final digit 0, 4, 5, or 9; and all those incapable of representing the sum of three square numbers. The remaining values of d will have for the final digit 1, 2, 3, 6, 7, or 8. Regarding these as separate cases, it is possible to devise a numerical test for each case. As an example, consider the values of d ending with $\cdots 3$. These must have the form $80k+3$ (sum of three odd biquadrates prime to 5) or the form $80k+33$ (sum of one odd and two even biquadrates prime to 5). Choosing the latter form,[2] seventy instances are found in a tabulation of values of d based on all admissible values of $x<700$. The first ten in order of magnitude are: 164833, 195313, 198593, 3029873, 4106193, 4590753, 5086913, 5693793, 5948193, 7424753. Testing these by a method (based upon the character of k), which need not be

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[1] Solutions of the congruence in question can be read from C. J. G. Jacobi's *Canon Arithmeticus*, Berlin, 1839, pp. 230–231.

[2] This form was considered first because it includes Norrie's solution, which appears in The University of St. Andrew 500th Anniversary Memorial Volume, Edinburgh, 1911, p. 89.

described in this brief discussion, it is found that the third integer is the sum of three biquadrates, and gives Norrie's solution

$$d = 198593 = \frac{353^4 - 272^4}{15^4} = 21^4 + 8^4 + 2^4.$$

The seventh integer furnishes a second numerical solution

$$d = 5086913 = \frac{651^4 - 599^4}{10^4} = 43^4 + 34^4 + 24^4.$$

It is the intention of the writer to carry the investigation of each case as far as values of $x < 1250$ $(= 2 \cdot 5^4)$. The fact that an additional solution was obtained in the first case examined would seem to give some promise that other solutions exist among the lower integers.

GLENSIDE, PA.

ON THE INVERSION OF THE q-SERIES ASSOCIATED WITH JACOBIAN ELLIPTIC FUNCTIONS

A. N. LOWAN, G. BLANCH AND W. HORENSTEIN[1]

The elliptic functions sn (u, k), cn (u, k) and dn (u, k) may be computed from theta functions by well known methods outlined in standard texts. (See, for instance, Whittaker & Watson, *Modern Analysis*, 4th edition, p. 485.) Given the modulus k, and $k' = (1-k^2)^{1/2}$, there is associated with k, k' a function ϵ defined by

$$\epsilon = \frac{1}{2}\left[\frac{1-(k')^{1/2}}{1+(k')^{1/2}}\right] = \frac{1}{2}\,\frac{\vartheta_2(0, q^4)}{\vartheta_3(0, q^4)}.$$

Values of theta functions for a given parameter q can be readily computed, and the Jacobi elliptic functions turn out to be ratios of the theta functions.

The series for ϵ in terms of q is given by

$$\epsilon = \sum_{k=0}^{\infty} q^{(2k+1)^2} \Big/ 1 + 2\sum_{k=1}^{\infty} q^{4k^2}. \tag{1}$$

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[1] The above results were obtained in the course of work carried out by the Mathematical Tables Project, conducted by the Work Projects Administration for the City of New York under the sponsorship of the National Bureau of Standards.

Since, however, the known parameter in elliptic functions is ϵ and not q, the problem of inverting formula (1) arises. Weierstrass[2] gave the first four terms of this expansion, and Milne-Thomson[3] added two more, giving

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + 1707\epsilon^{17} + 20,910\epsilon^{21} + \cdots .$$

As is well known, the above series converges for $|\epsilon| < 1/2$; and it is rapidly convergent over a large range of ϵ. But when ϵ is greater than 0.3, only a few decimals in q may be obtained from the six terms given above. Because of the importance of this series in the computation of elliptic functions as well as in the Jacobian zeta functions, the above series has been extended to fourteen terms, as given below:

$$\begin{aligned} q = \epsilon[&1 + 2u + 15u^2 + 150u^3 + 1707u^4 + 20,910u^5 + 268,616u^6 \\ &+ 3,567,400u^7 + 48,555,069u^8 + 673,458,874u^9 \\ &+ 9,481,557,398u^{10} + 135,119,529,972u^{11} \\ &+ 1,944,997,539,623u^{12} + 28,235,172,753,886u^{13} \cdots] \end{aligned} \tag{2}$$

where $u = \epsilon^4$.

The following accuracy may be obtained with the above series:

ϵ	Accuracy in q with 14 terms of (2)	Accuracy in q with 6 terms of (2)
	decimals	*decimals*
.4	7	4
.36	10	5
.30	14	6
.25	18	8

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[2] *Werke*, vol. 2 (1895), p. 276.

[3] *Ten decimal table of the nome q*, Journal of the London Mathematical Society, vol. 5 (1930) pp. 148–149.

TABLE OF THE ZEROS OF THE LEGENDRE POLYNOMIALS OF ORDER 1–16 AND THE WEIGHT COEFFICIENTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA[1]

ARNOLD N. LOWAN, NORMAN DAVIDS AND ARTHUR LEVENSON

Gauss' method of mechanical quadrature has the advantage over most methods of numerical integration in that it requires about half the number of ordinate computations. This is desirable when such computations are very laborious, or when the observations necessary to determine the average value of a continuously varying physical quantity are very costly. Gauss' classical result[2] states that, for the range $(-1, +1)$, the "best" accuracy with n ordinates is obtained by choosing the corresponding abscissae at the zeros $x_1, \cdots, x_n$ of the Legendre polynomials $P_n(x)$. With each x_i is associated a constant a_i such that

$$\int_{-1}^{1} f(x)dx \sim a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n). \tag{1}$$

The accompanying table computed by the Mathematical Tables Project gives the roots x_i for each $P_n(x)$ up to $n=16$, and the corresponding weight coefficients a_i, to 15 decimal places.

The first such table, computed by Gauss gave 16 places up to $n=7$.[3] More recently work was done by Nyström,[4] who gave 7 decimals up to $n=10$, but for the interval $(-1/2, +1/2)$. B. de F. Bayly has given the roots and coefficients of $P_{12}(x)$ to 13 places.[5]

The Gaussian quadrature formula for evaluating an integral with arbitrary limits (p, q) is given by

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[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project conducted by the Work Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

[2] *Methodus nova integralium valores per approximationen inveniendi*, Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores, vol. 3 (1814), or *Werke*, vol. 3, pp. 193–195.

[3] It may be found reproduced in Heine's *Kugelfunctionen*, vol. 2, 1881, p. 15, or Hobson, *Spherical Harmonics*, pp. 80–81.

[4] Nyström, Acta Mathematica, vol. 54 (1930), p. 191.

[5] B. de F. Bayly, Biometrika, vol. 30 (1938), pp. 193–194.

$$(2) \qquad \int_p^q f(x)dx = \frac{q-p}{2}\sum_{i=1}^{n} a_i f\left(x_i\frac{q-p}{2}+\frac{q+p}{2}\right)+R_n(f)$$

where $x_i=x_{i,n}$ is the ith root of $P_n(x)$ and

$$(3) \qquad a_i = a_{i,n} = \frac{1}{P_n'(x_i)}\int_{-1}^{1}\frac{P_n(x)}{x-x_i}dx.$$

We have

$$x_i=-x_{n-i+1},\ a_i=a_{n-i+1},$$

so that only half the values need be tabulated.

It is well known that if $f_{2n-1}(x)$ is an arbitrary polynomial of degree at most $2n-1$, then

$$R_n(f_{2n-1}) = 0,$$

that is, formula (1) is exact. Thus Gauss' formula with n ordinates provides an approximation as good as would have been obtained by using a polynomial of degree $2n-1$.

If $f(x)$ has a continuous derivative of order $2n$ in the interval (p, q), then[6]

$$R_n = \frac{f^{(2n)}(\xi)}{(2n)!k_n^2}$$

where ξ is a point in the interval (p, q) and k_n is the normalizing factor for $P_n(x)$, which is equal to

$$\left(\frac{2n+1}{q-p}\right)^{1/2}$$

The roots were calculated by successive approximations, combining synthetic division with Newton's tangent formula. They were checked by using the relations between roots and coefficients.

By suitable transformations of formula (3), the computation of the weight coefficients a_i was made to depend on that of the roots, in particular on the values of the successive quotients obtained in the process of synthetic division. These weight coefficients were checked by putting $f(x)\equiv 1$ in formula (1), giving

$$2 = a_1 + a_2 + \cdots + a_n.$$

The sum of the a_i's was required not to differ from 2 by more than one unit in the 17th place.

[6] This expression is due to Markoff. See Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, p. 369.

TABLE

$n=2$	
$x_1=0.577350269189626$	$a_1=1.000000000000000$
$n=3$	
$x_0=0.000000000000000$	$a_0=0.888888888888889$
$x_1=0.774596669241483$	$a_1=0.555555555555556$
$n=4$	
$x_1=0.339981043584856$	$a_1=0.652145154862546$
$x_2=0.861136311594053$	$a_2=0.347854845137454$
$n=5$	
$x_0=0.000000000000000$	$a_0=0.568888888888889$
$x_1=0.538469310105683$	$a_1=0.478628670499366$
$x_2=0.906179845938664$	$a_2=0.236926885056189$
$n=6$	
$x_1=0.238619186083197$	$a_1=0.467913934572691$
$x_2=0.661209386466265$	$a_2=0.360761573048139$
$x_3=0.932469514203152$	$a_3=0.171324492379170$
$n=7$	
$x_0=0.000000000000000$	$a_0=0.417959183673469$
$x_1=0.405845151377397$	$a_1=0.381830050505119$
$x_2=0.741531185599394$	$a_2=0.279705391489277$
$x_3=0.949107912342759$	$a_3=0.129484966168870$
$n=8$	
$x_1=0.183434642495650$	$a_1=0.362683783378362$
$x_2=0.525532409916329$	$a_2=0.313706645877887$
$x_3=0.796666477413627$	$a_3=0.222381034453374$
$x_4=0.960289856497536$	$a_4=0.101228536290376$
$n=9$	
$x_0=0.000000000000000$	$a_0=0.330239355001260$
$x_1=0.324253423403809$	$a_1=0.312347077040003$
$x_2=0.613371432700590$	$a_2=0.260610696402935$
$x_3=0.836031107326636$	$a_3=0.180648160694857$
$x_4=0.968160239507626$	$a_4=0.081274388361574$
$n=10$	
$x_1=0.148874338981631$	$a_1=0.295524224714753$
$x_2=0.433395394129247$	$a_2=0.269266719309996$
$x_3=0.679409568299024$	$a_3=0.219086362515982$
$x_4=0.865063366688985$	$a_4=0.149451349150581$
$x_5=0.973906528517172$	$a_5=0.066671344308688$
$n=11$	
$x_0=0.000000000000000$	$a_0=0.272925086777901$
$x_1=0.269543155952345$	$a_1=0.262804544510247$
$x_2=0.519096129110681$	$a_2=0.233193764591990$
$x_3=0.730152005574049$	$a_3=0.186290210927734$
$x_4=0.887062599768095$	$a_4=0.125580369464905$
$x_5=0.978228658146057$	$a_5=0.055668567116174$

TABLE—*Continued*

$n=12$	
$x_1=0.125333408511469$	$a_1=0.249147045813403$
$x_2=0.367831498918180$	$a_2=0.233492536538355$
$x_3=0.587317954286617$	$a_3=0.203167426723066$
$x_4=0.769902674194305$	$a_4=0.160078328543346$
$x_5=0.904117256370475$	$a_5=0.106939325995318$
$x_6=0.981560634246719$	$a_6=0.047175336386512$
$n=13$	
$x_0=0.000000000000000$	$a_0=0.232551553230874$
$x_1=0.230458315955135$	$a_1=0.226283180262897$
$x_2=0.448492751036447$	$a_2=0.207816047536889$
$x_3=0.642349339440340$	$a_3=0.178145980761946$
$x_4=0.801578090733310$	$a_4=0.138873510219787$
$x_5=0.917598399222978$	$a_5=0.092121499837728$
$x_6=0.984183054718588$	$a_6=0.040484004765316$
$n=14$	
$x_1=0.108054948707344$	$a_1=0.215263853463158$
$x_2=0.319112368927890$	$a_2=0.205198463721296$
$x_3=0.515248636358154$	$a_3=0.185538397477938$
$x_4=0.687292904811685$	$a_4=0.157203167158194$
$x_5=0.827201315069765$	$a_5=0.121518570687903$
$x_6=0.928434883663574$	$a_6=0.080158087159760$
$x_7=0.986283808696812$	$a_7=0.035119460331752$
$n=15$	
$x_0=0.000000000000000$	$a_0=0.202578241925561$
$x_1=0.201194093997435$	$a_1=0.198431485327111$
$x_2=0.394151347077563$	$a_2=0.186161000015562$
$x_3=0.570972172608539$	$a_3=0.166269205816994$
$x_4=0.724417731360170$	$a_4=0.139570677926154$
$x_5=0.848206583410427$	$a_5=0.107159220467172$
$x_6=0.937273392400706$	$a_6=0.070366047488108$
$x_7=0.987992518020485$	$a_7=0.030753241996117$
$n=16$	
$x_1=0.095012509837637$	$a_1=0.189450610455069$
$x_2=0.281603550779259$	$a_2=0.182603415044924$
$x_3=0.458016777657227$	$a_3=0.169156519395003$
$x_4=0.617876244402644$	$a_4=0.149595988816577$
$x_5=0.755404408355003$	$a_5=0.124628971255534$
$x_6=0.865631202387832$	$a_6=0.095158511682493$
$x_7=0.944575023073233$	$a_7=0.062253523938648$
$x_8=0.989400934991650$	$a_8=0.027152459411754$

EXAMPLE. Let it be desired to find $I=\int_1^2 dx/x$. By formula (2)

$$I \sim \sum \frac{a_k}{x_k^3}.$$

Taking $n=10$, we have

$$I \sim \frac{0.295524224714753}{2.851125661018369} + \frac{0.269266719309996}{2.566604605870753}$$
$$+ \frac{0.219086362515982}{2.320590431700976} + \frac{0.149451349150581}{2.134936633311015}$$
$$+ \frac{0.066671344308688}{2.026093471482828} + \frac{0.295524224714753}{3.148874338981631}$$
$$+ \frac{0.269266719309996}{3.433395394129247} + \frac{0.219086362515982}{3.679409568299024}$$
$$+ \frac{0.149451349150581}{3.865063366688985} + \frac{0.066671344308688}{3.973906528517172}$$
$$= 0.693147180559947.$$

This agrees to fourteen places with the actual value

$$\log 2 = 0.693147180559945.$$

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NEW YORK CITY

SOME REMARKS ON RELATIVELY COMPLETE FIELDS

IRVING KAPLANSKY AND O. F. G. SCHILLING

In this note we discuss some properties of fields which are relatively complete with respect to valuations of rank one. We show that relatively complete fields have in many respects simpler properties than complete fields. This note arose as a consequence of a discussion of an erroneous statement in a note of one of the authors.[1]

Let K be a field on which a valuation V of rank one is defined. Denote the ring of integers in K with respect to V by O and let P be the prime ideal of nonunits in O.

DEFINITION[2] 1. *The field K is termed relatively complete with respect to V if every polynomial congruence*

$$f(x) \equiv g_0(x)h_0(x) \pmod{P}$$

with $f(x)$ in $O[x]$ and $(g_0(x), h_0(x)) \equiv 1 \pmod{P}$ implies

$$f(x) = g(x)h(x)$$

where $g(x) \equiv g_0(x) \pmod{P}$ and $h(x) \equiv h_0(x) \pmod{P}$.

We remark[3] that there exist relatively complete fields which are not complete with respect to the congruence topology induced by V. To find examples for such fields it suffices to consider infinite algebraic extensions of a field which is complete with respect to a discrete valuation of rank one.

DEFINITION[4] 2. *Two polynomials $a(x)$ and $b(x)$ over a field K are said to have the same decomposition type if $a(x) = \prod_{i=1}^{s} p_i(x)^{m_i}$, $b(x) = \prod_{i=1}^{r} q_i(x)^{n_i}$ when the $p_i(x)$ and $q_i(x)$ are irreducible polynomials and $s = r$, $m_i = n_i$ and $p_i(x)$ and $q_i(x)$ have the same degree for a suitable arrangement of the prime factors.*

We are now able to generalize certain of the results of F. K.

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[1] O. F. G. Schilling, *Remarks on a special class of algebras*, American Journal of Mathematics, vol. 62 (1940), pp. 346–352.

[2] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Mathematische Zeitschrift, vol. 39 (1935), pp. 269–404.

[3] A. Ostrowski, *Über einige Fragen der allgemeinen Körpertheorie*, Journal für die reine und angewandte Mathematik, vol. 143 (1913), pp. 255–284.

[4] F. K. Schmidt, *Mehrfach perfekte Körper*, Mathematische Annalen, vol. 108 (1933), pp. 1–25.

Schmidt. We first state without proof a simple consequence of Ostrowski's criterion for relative completeness.[5]

LEMMA 1. *Let K be a field which is relatively complete with respect to a rank one valuation V and let $\overline{K}$ be the completion of K. Then the decomposition type of any separable polynomial $f(x)$ with coefficients in K is the same in K as in $\overline{K}$.*

LEMMA[6] 2. *Let $\overline{K}$, $\overline{K}_1$ be the completions of K with respect to the inequivalent rank one valuations V, V_1 of K, and let the fields L, L_1 be relatively complete with respect to V and V_1 such that $K \leqq L \leqq \overline{K}$, $K \leqq L_1 \leqq \overline{K}_1$. Then there exists for any two separable polynomials $g(x)$, $g_1(x)$ of the same degree in L, L_1 a third polynomial $h(x)$ in K which has the same decomposition types as $g(x)$, $g_1(x)$ in L, L_1.*

PROOF. According to a result of F. K. Schmidt[7] there exists a polynomial $h(x)$ in K whose decomposition types in $\overline{K}$, $\overline{K}_1$ coincide with those of $g(x)$, $g_1(x)$. Then, by Lemma 1, the decomposition types of $h(x)$ and $g(x)$ are the same in L. Similarly, the decomposition types of $h(x)$ and $g_1(x)$ are the same in L_1.

Next we prove the following theorem.[8]

THEOREM 1. *If the field K is relatively complete with respect to the rank one valuation V and if, for the inequivalent rank one valuation V_1, the field L_1 is a relatively complete field containing K and contained in the completion of K, then L_1 is separably algebraically closed.*

PROOF. Let $g_1(x)$ be an irreducible separable polynomial of degree n in L_1 and let $g(x)$ be the product of any n distinct linear factors in K. Then, according to Lemma 2, the polynomials $g(x)$ and $g_1(x)$ have the same type of decomposition in K and L_1. Hence $g_1(x)$ must be a product of linear polynomials; in other words, the field L_1 is separably algebraically closed.

THEOREM 2. *A field K is relatively complete with respect to two inequivalent rank one valuations if and only if it is separably algebraically closed.*

PROOF. The necessity of the statement follows at once from Theorem 1. On the other hand Ostrowski's criterion implies that a separa-

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[5] Ostrowski's criterion states that a field is relatively complete if and only if it is separably algebraically closed in its complete closure. Loc. cit., p. 316.

[6] F. K. Schmidt, loc. cit., lemma on p. 16.

[7] Loc. cit., p. 16.

[8] Theorem 1 is the analogue of Schmidt's Theorem 1, p. 18, loc. cit.

bly algebraically closed field is relatively complete with respect to every valuation of rank one.

THEOREM 3. *If a field K is complete with respect to a rank one valuation and relatively complete with respect to a rank one valuation which is inequivalent to the first valuation, then K is a multiply complete field.*

PROOF. By Theorem 2 the field K is separably algebraically closed. Since K is complete with respect to the second valuation, it follows that K is algebraically closed.[9] Moreover the cardinal number $\mathfrak{c}$ of K satisfies the equation $\mathfrak{c}^{\aleph_0}=\mathfrak{c}$ for K is a complete field. Hence K is a multiply complete field.

We should like to point out that the above theorems do not require conditions on the cardinal number of the field K as in the theory of F. K. Schmidt for complete fields.

In Theorem 2 it was proved that the separable algebraic closure of a field K is always relatively complete with respect to every valuation extending any non-trivial rank one valuation of K. We then may ask whether already a finite algebraic extension L of K is relatively complete with respect to some rank one valuation though K itself is not relatively complete with respect to any rank one valuation.

We shall give a partial answer to this problem in this theorem.

THEOREM 4. *Suppose that K is a field which has for every integer n exactly one separable extension of degree n and suppose that K is not relatively complete with respect to any non-trivial rank one valuation. Then there exists no finite algebraic extension L of K which is relatively complete with respect to a rank one valuation.*

PROOF. Suppose that, contrary to the theorem, there exists a finite extension L of degree m over K which is relatively complete with respect to some rank one valuation V_L. Since L is relatively complete with respect to V_L there is, by Ostrowski's criterion, a one-to-one correspondence between the algebraic extensions of L and its completion $\overline{L}$ with respect to V_L. In other words, if Z_n is an arbitrary extension of degree n over L then $[Z_n\overline{L}:\overline{L}]=n$. Consider next the valuation V on K which is induced by V_L. Let $\overline{K}$ be the completion of K with respect to V. We assert that the Galois group of the algebraic closure $\overline{K}'$ of $\overline{K}$ (relative to $\overline{K}$) is cyclic. For a proof it suffices to show that the decomposition groups for the prolongations V_i of V to any extension $M>K$ are cyclic.[10] Since M/K is cyclic all decom-

[9] F. K. Schmidt, loc. cit. Lemma 2, p. 18.

[10] M. Deuring, *Verzweigungstheorie bewerteter Körper*, Mathematische Annalen, vol. 105 (1931), pp. 277–307.

position fields coincide and are cyclic. The field $\bar{L}$ is then equivalent to a subfield of $\bar{K}'$; without loss of generality we may suppose $\bar{K}' > \bar{L} \geqq \bar{K}$. The degree $[\bar{L}:\bar{K}] = \bar{m}$ is a divisor of m. Consequently $[Z_n\bar{L}:\bar{K}] = [Z_n\bar{L}:\bar{L}][\bar{L}:\bar{K}] = n\bar{m}$. By the Galois theory there is then for every integer n an extension Z_n^* of degree n over $\bar{K}$. The defining equation $f^*(x) = 0$ of $Z_n^*/\bar{K}$ now may be approximated by an irreducible equation $f(x) = 0$ of degree n with coefficients in K so that Z_n^* is generated by the roots of $f(x) = 0$. The root field of $f(x) = 0$ over K is the cyclic extension Z_n' of degree n over K. Hence $Z_n^* = Z_n'\bar{K}$ for all n, contrary to the assumption that K is not relatively complete with respect to any rank one valuation.

HARVARD UNIVERSITY AND
UNIVERSITY OF CHICAGO

A DIFFERENTIAL GEOMETRY PROBLEM USING TENSOR ANALYSIS

ATHERTON H. SPRAGUE

1. **Introduction.** The problem at hand was worked out in attempting to apply tensors to a much more general problem in classical differential geometry. The results obtained in a general coordinate system reduce readily to classical results of Eisenhart. An interesting interpretation of Christoffel symbols appears.

2. **R net.** A rectilinear congruence in 3-space is called a W-congruence if the asymptotic lines on the two focal surfaces correspond. If the tangents to both families of curves of a conjugate net on a surface form W-congruences the net is called an R net.[1] We derive the analytic conditions that must obtain in order that a given conjugate net on a surface shall be an R net.

3. **Equations for an R net.** Let S_1 be one focal surface of a W-congruence, the vector equation of the surface being

$$z_1^\alpha = z_1^\alpha(x^i), \qquad \alpha = 1, 2, 3; \; i = 1, 2. \tag{3.1}$$

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[1] Tzitzeica, Comptes Rendus de l'Académie des Sciences, Paris, vol. 152 (1911), p. 1077.

Let S_2 be the other focal surface of the congruence with vector z_2^α so that we have

$$z_2^\alpha = z_1^\alpha + \rho_1 \xi_1^\alpha \tag{3.2}$$

where ξ_1^α is a unit vector tangent to S_1, and ρ_1 is an invariant.

Then if $\lambda_{1/}^i$ are the components of this vector in the x's we have

$$\xi_1^\alpha = z_{1/,i}^\alpha \lambda_{1/}^i \tag{3.3}$$

where in this case

$$z_{1/,i}^\alpha = \frac{\partial z_1^\alpha}{\partial x^i}$$

since z_1^α being an invariant for a transformation of coordinates in the x's, the ordinary derivative of z_1^α with respect to x^i is the same as the covariant derivative with respect to g_{ij}, the fundamental tensor of S_1. Then substituting (3.3) in (3.2) we have

$$z_2^\alpha = z_1^\alpha + \rho_1 z_{1/,i}^\alpha \lambda_{1/}^i. \tag{3.4}$$

Similarly by the property of focal surfaces we have

$$z_1^\alpha = z_2^\alpha + \rho_2 z_{2/,i}^\alpha \lambda_{2/}^i \tag{3.5}$$

where ρ_2 is an invariant, and $\lambda_{2/}^i$ is a unit vector tangent to S_2.

Adding (3.4) and (3.5) we have

$$\rho_1 \lambda_{1/}^i z_{1/,i}^\alpha + \rho_2 \lambda_{2/}^i z_{2/,i}^\alpha = 0. \tag{3.6}$$

From (3.6) we have

$$\rho_1 = \bar{e}\rho_2, \tag{3.6'}$$

$$\lambda_{1/}^i z_{1/,i}^\alpha = e\lambda_{2/}^i z_{2/,i}^\alpha$$

where $\bar{e}=1$ if $e=-1$, and $\bar{e}=-1$ if $e=1$, and conversely.

We differentiate (3.4) covariantly and have

$$z_{2/,k}^\alpha = z_{1/,k}^\alpha + \rho_{1/,k}\lambda_{1/}^i z_{1/,i}^\alpha + \rho_1 \lambda_{1/,k}^i z_{1/,i}^\alpha + \rho_1 \lambda_{1/}^i z_{1/,ik}^\alpha. \tag{3.7}$$

Let η_1^α be the unit normal to S_1.

We multiply (3.7) by $\lambda_{2/}^k$, sum for k, multiply by η_1^α, sum for α and we have

$$0 = \rho_1 b_{1/ij} \lambda_{1/}^i \lambda_{2/}^j \tag{3.8}$$

the first three terms on the right vanishing because η_1^α is perpendicular to S_1, the term on the left vanishing because of this fact and (3.6).

and the last term becoming $\rho_1\eta_1^\alpha \cdot z^\alpha_{1/,ij}\lambda^i_{1/}\lambda^j_{2/}$ the second factor of which is denoted by $b_{1/ij}\lambda^i_{1/}\lambda^j_{2/}$.[2]

Hence *the directions* $\lambda^i_{1/}$ *and* $\lambda^j_{2/}$ *are conjugate on* S_1.[3]

Similarly they are conjugate on S_2.

We next differentiate (3.7) covariantly with respect to g_{ij}, the fundamental tensor of S_1,

$$
\begin{aligned}
\frac{\partial^2 z_2^\alpha}{\partial x^k \partial x^j} - z^\alpha_{2/,m}\begin{Bmatrix} m \\ kj \end{Bmatrix}_{ij} &= z^\alpha_{1/,kj} + \rho_{1/,kj}\lambda^i_{1/}z^\alpha_{1/,i} + \rho_{1/,k}\lambda^i_{1/,j}z^\alpha_{1/,i} \\
&+ \rho_{1/,k}\lambda^i_{1/}z^\alpha_{1/,ij} + \rho_{1/,j}\lambda^i_{1/,k}z^\alpha_{1/,i} + \rho_1\lambda^i_{1/,kj}z^\alpha_{1/,i} + \rho_1\lambda^i_{1/,k}z^\alpha_{1/,ij} \\
&+ \rho_{1/,j}\lambda^i_{1/}z^\alpha_{1/,ik} + \rho_1\lambda^i_{1/,j}z^\alpha_{1/,ik} + \rho_1\lambda^i_{1/}z^\alpha_{1/,ikj}.
\end{aligned} \tag{3.9}
$$

Multiply by η_2^α (the unit normal to S_2), sum for α and we have

$$
\begin{aligned}
b_{2/kj} &= b_{1/kj}\eta^\alpha_1 \cdot \eta^\alpha_2 + \rho_{1/,kj}\lambda^i_{1/}z^\alpha_{1/,i} \cdot \eta^\alpha_2 + \rho_{1/,k}\lambda^i_{1/,j}z^\alpha_{1/,i} \cdot \eta^\alpha_2 \\
&+ \rho_{1/,k}\lambda^i_{1/}b_{1/ij}\eta^\alpha_1 \cdot \eta^\alpha_2 + \rho_{1/,j}\lambda^i_{1/,k}z^\alpha_{1/,i} \cdot \eta^\alpha_2 + \rho_1\lambda^i_{1/,kj}z^\alpha_{1/,i} \cdot \eta^\alpha_2 \\
&+ \rho_1\lambda^i_{1/,k}b_{1/ij}\eta^\alpha_1 \cdot \eta^\alpha_2 + \rho_{1/,j}\lambda^i_{1/}b_{1/ik}\eta^\alpha_1 \cdot \eta^\alpha_2 + \rho_1\lambda^i_{1/,j}b_{1/ik}\eta^\alpha_1 \cdot \eta^\alpha_2 \\
&+ \rho_1\lambda^i_{1/}b_{1/ik,j}\eta^\alpha_1 \cdot \eta^\alpha_2 + b_{1/ik}\rho_1\lambda^i_{1/}\eta^\alpha_{1/,j} \cdot \eta^\alpha_2.
\end{aligned} \tag{3.10}
$$

In the future, since unless otherwise stated covariant differentiation is with respect to the fundamental tensor of S_1, we shall note the covariant derivative of $z^\alpha_{2/,k}$ by $z^\alpha_{2/kj}$.

We evaluate $z^\alpha_{1/,i} \cdot \eta^\alpha_2$ as follows.

We differentiate (3.5) covariantly giving

$$
z^\alpha_{1/,i} = z^\alpha_{2/,i} + \rho_{2/,i}\lambda^s_{2/}z^\alpha_{2/,s} + \rho_2\lambda^s_{2/,i}z^\alpha_{2/,s} + \rho_2\lambda^s_{2/}z^\alpha_{2/,si}. \tag{3.11}
$$

Multiply by η_2^α and sum for α, giving

$$
z^\alpha_{1/,i} \cdot \eta^\alpha_2 = \rho_2\lambda^s_{2/}b_{2/si}. \tag{3.12}
$$

Substituting this value for $z^\alpha_{1/,i} \cdot \eta^\alpha_2$ in (3.10) we have

$$
\begin{aligned}
b_{2/kj} &= (\eta^\alpha_1 \cdot \eta^\alpha_2)(b_{1/kj} + b_{1/ij}\rho_{1/,k}\lambda^i_{1/} + b_{1/ij}\rho_1\lambda^i_{1/,k} + b_{1/ik}\rho_{1/,j}\lambda^i_{1/} \\
&+ b_{1/ik}\rho_1\lambda^i_{1/,j} + b_{1/ik,j}\rho_1\lambda^i_{1/}) + \rho_1\lambda^i_{1/}b_{1/ik}\eta^\alpha_{1/,j} \cdot \eta^\alpha_2 \\
&+ \rho_{1/,k}\lambda^i_{1/,j}\rho_2\lambda^s_{2/}b_{2/si} + \rho_{1/,j}\lambda^i_{1/,k}\rho_2\lambda^s_{2/}b_{2/si} \\
&+ \rho_1\lambda^i_{1/,kj}\rho_2\lambda^s_{2/}b_{2/si}.
\end{aligned} \tag{3.13}
$$

Since the asymptotic lines on S_1 and S_2 are to correspond we have

[2] L. P. Eisenhart, *Riemannian Geometry*, 1926, Equation 56.2, p. 189.

[3] Ibid., Equation 56.3, p. 189.

$$b_{2/i\alpha} = \mu b_{1/i\alpha}. \tag{3.14}$$

We determine μ as follows. Differentiate the second of (3.6′) covariantly, giving

$$\lambda^{i}_{1/,j}z^{\alpha}_{1/,i} + \lambda^{i}_{1/}z^{\alpha}_{1/,ij} + e(\lambda^{i}_{2/,j}z^{\alpha}_{2/,i} + \lambda^{i}_{2/}z^{\alpha}_{2/,ij}) = 0. \tag{3.15}$$

Multiply by η^{α}_{2}, sum for α and use (3.12). This becomes

$$\lambda^{i}_{1/}b_{1/ij}(\eta^{\alpha}_{1}\cdot\eta^{\alpha}_{2}) + \rho_2\lambda^{i}_{1/,j}\lambda^{k}_{2/}b_{2/ki} + e\lambda^{i}_{2/}b_{2/ij} = 0. \tag{3.16}$$

Now multiply by $\lambda^{j}_{1/}$ and sum for j giving

$$(\eta^{\alpha}_{1}\cdot\eta^{\alpha}_{2})b_{1/ij}\lambda^{i}_{1/}\lambda^{j}_{1/} + b_{2/ik}\rho_2\lambda^{i}_{1/,j}\lambda^{k}_{2/}\lambda^{j}_{1/} = 0. \tag{3.17}$$

But since $b_{2/ik} = \mu b_{1/ik}$ we have

$$\mu = -\frac{(\eta^{\alpha}_{1}\cdot\eta^{\alpha}_{2})b_{1/ij}\lambda^{i}_{1/}\lambda^{j}_{1/}}{\rho_2 b_{1/ik}\lambda^{i}_{1/,j}\lambda^{k}_{2/}\lambda^{j}_{1/}}. \tag{3.18}$$

Substituting in (3.13) and using the fact that[3]

$$\eta^{\alpha}_{1/,j} = -b_{1/ej}g^{em}z^{\alpha}_{1/,m}$$

we have

$$\begin{aligned}
&\rho_2 b_{i\alpha}\lambda^{\alpha}_{2/}\lambda^{i}_{1/,k}\lambda^{k}_{1/}[b_{kj} + \rho_{1/,k}\lambda^{i}_{1/}b_{ij} + \rho_1\lambda^{i}_{1/,k}b_{ij} + \rho_{1/,j}\lambda^{i}_{1/}b_{ik} \\
&+ \rho_1\lambda^{i}_{1/,j}b_{ik} + \rho_1\lambda^{i}_{1/}b_{ik,j}] + b_{kr}\lambda^{k}_{1/}\lambda^{r}_{1/}[b_{kj} - \rho_{1/,k}\lambda^{i}_{1/,j}\rho_2\lambda^{\alpha}_{2/}b_{i\alpha} \\
&- \rho_1\rho_2\lambda^{i}_{1/,kj}\lambda^{\alpha}_{2/}b_{i\alpha} + b_{ik}\rho_1\rho_2\lambda^{i}_{1/}b_{ej}g^{em}b_{m\alpha}\lambda^{\alpha}_{2/} \\
&- \rho_{1/,j}\lambda^{i}_{1/,k}\rho_2\lambda^{\alpha}_{2/}b_{i\alpha}] = 0
\end{aligned} \tag{3.19}$$

where the b's are all those of S_1.

To evaluate ρ_1, multiply (3.16) by $\lambda^{j}_{2/}$ and sum for j, giving, by (3.14)

$$\rho_1 b_{1/ki}\lambda^{k}_{2/}\lambda^{i}_{1/,j}\lambda^{j}_{2/} + \lambda^{i}_{2/}\lambda^{j}_{2/}b_{1/ij} = 0. \tag{3.20}$$

Similarly, we have

$$\rho_2 b_{1/ki}\lambda^{k}_{1/}\lambda^{i}_{2/,j}\lambda^{j}_{1/} + \lambda^{i}_{1/}\lambda^{j}_{1/}b_{1/ij} = 0 \tag{3.21}$$

where $\lambda^{i}_{2/,j}$ is with respect to $\bar{g}_{ij}$ of S_2. (It should be remarked that ρ_2 may be expressed entirely in terms of elements of S_1 by means by (3.6) and (3.7) and differentiation.)

Equations (3.19) with ρ_1 and ρ_2 determined by (3.20) and (3.21), respectively, constitute the condition that must obtain in order for

the tangents to the curves of direction $\lambda^i_{1/}$ on S_1 to form a W-congruence. An equation similar to (3.19) obtains for the direction $\lambda^j_{2/}$. These two equations must hold in order for the net with directions $\lambda^i_{1/}$ and $\lambda^j_{2/}$ to be an R net.

In particular we consider the case where $\lambda^i_{1/}$ and $\lambda^j_{2/}$ are tangent to the u and v parametric curves, respectively. Then[4]

$$\begin{aligned} \lambda^1_1 &= 1, & b_{11} &= D, \\ \lambda^2_1 &= 0, & b_{12} &= b_{21} = D' = 0, \\ \lambda^1_2 &= 0, & b_{22} &= D'', \\ \lambda^2_2 &= 1, \end{aligned} \tag{3.22}$$

and it is easily shown that

$$\lambda^i_{aj.i} = \left\{ \begin{matrix} i \\ aj \end{matrix} \right\}$$

for any *fixed* i, a, j, which is an interesting interpretation of the Christoffel symbols in this case.

In this case (3.19) reduces to

$$2\frac{\partial}{\partial v}\left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \frac{\partial}{\partial u}\left(\left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} - \frac{D''}{D}\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \right), \tag{3.23}$$

the equation obtained by Eisenhart.[5]

The equation similar to (3.19) reduces to

$$2\frac{\partial}{\partial u}\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = \frac{\partial}{\partial v}\left(\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} - \frac{D}{D''}\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \right) \tag{3.24}$$

and these two equations constitute the condition that the parametric curves of a surface S form an R net.

AMHERST COLLEGE

[4] L. P. Eisenhart, *Differential Geometry*, 1909, p. 115.

[5] L. P. Eisenhart, *Transformation of Surfaces*, 1923, p. 106.

ON THE NILPOTENCY OF THE RADICAL OF A RING

RICHARD BRAUER[1]

1. **Introduction.** A few years ago, it was shown by C. Hopkins[2] that the structure theory of noncommutative rings[3] can be based on the assumption of only the minimum condition for left-ideals. Before Hopkins, a maximum condition for ideals had also been used in order to prove that the radical of the ring is nilpotent. Actually this last fact is a special case of the maximum condition, for example, the existence of a maximal nilpotent (two-sided) ideal, and this makes Hopkins' result appear rather surprising.

In this note, I give a short and simple proof for Hopkins' theorem. I also show that it is sufficient to assume only the minimum condition for sets of two-sided nil-ideals (that is, ideals consisting only of nilpotent elements) in order to prove the nilpotency of the radical. The later sections are concerned with the existence of idempotents and primitive left-ideals contained in a given regular left-ideal. Here the assumptions concerning the ring R are those on which Köthe[4] and Deuring[5] based their treatment of noncommutative rings. As was shown by Köthe, these assumptions are equivalent to the validity of the structure theory, so that it is natural to work with them. Once the results of the later sections have been established, there is no difficulty in developing the theory with the usual methods.[6]

2. **Preliminaries.** A ring R is a set of elements for which an addition and a multiplication are defined such that the elements form an abelian group under addition and that the associative law of multiplication and both distributive laws hold. We may also have a set K of operators. Then the product $t\alpha = \alpha t$ of any α in R with any t in K must be defined as an element of R, and the following rules are to hold (α, β in R, t in K)

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[1] Guggenheim Fellow.

[2] Charles Hopkins, Duke Mathematical Journal, vol. 4 (1938), p. 664; cf. also J. Levitzki, Compositio Mathematica, vol. 7 (1939), p. 214.

[3] E. Artin, Hamburger Abhandlungen, vol. 5 (1928), p. 251; B. L. van der Waerden, *Moderne Algebra*, vol. 2; M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939.

[4] G. Köthe, Mathematische Zeitschrift, vol. 32 (1930), p. 161.

[5] Loc. cit.

[6] The treatment thus obtained seems to me simpler than Deuring's treatment.

$$(1) \qquad (\alpha+\beta)t=\alpha t+\beta t, \qquad (\alpha\beta)t=\alpha(\beta t)=(\alpha t)\beta.$$

We then say that R is a K-ring. However, for some purposes, these postulates are not suitable, for example, it is easy to see that it is not always possible to imbed a K-ring R in a K-ring R^* which has a 1-element. We may modify the definition of a K-ring R in the following manner: If t lies in K and α lies in R, then αt and $t\alpha$ both are defined as elements of R. For α, β in R, and for t in K, we have

$$(2) \qquad \begin{aligned} &(\alpha+\beta)t=\alpha t+\beta t, \qquad t(\alpha+\beta)=t\alpha+t\beta,\\ &(\alpha\beta)t=\alpha(\beta t), \qquad (\alpha t)\beta=\alpha(t\beta), \qquad t(\alpha\beta)=t(\alpha\beta).\end{aligned}$$

We admit the possibility that $\alpha t\neq t\alpha$. A K-ring R in this sense can always be imbedded in a K-ring R^* which has a 1-element. It does not mean an essential restriction to assume that K itself is a ring which has a 1-element e such that: (a) $\alpha e=e\alpha=\alpha$ for all α in R. (b) If $\alpha t=0$ for a fixed t in K and all α in R, then $t=0$. The same holds, if all $t\alpha=0$. (c) For the elements of R and for the elements of K, all possible associative and distributive laws hold. (This includes the equations (2).) A left-ideal (abbreviated l-ideal) of the K-ring R is a subset $\mathfrak{a}$ of R which satisfies the following conditions: (1) If α and β lie in $\mathfrak{a}$, then $\alpha\pm\beta$ lies in $\mathfrak{a}$. (2) If α lies in $\mathfrak{a}$, then $\rho\alpha$ and $t\alpha$ lie in $\mathfrak{a}$ for any ρ in R and any t in K. In the case of a right-ideal (r-ideal), (2) has to be replaced by: (2′) If α lies in $\mathfrak{a}$, then $\alpha\rho$ and αt lie in $\mathfrak{a}$ for any ρ in R and any t in K. A set $\mathfrak{a}$ is an ideal, if $\mathfrak{a}$ is both l-ideal and r-ideal.

For the following, it does not make any difference which definition of a K-ring is used.

3. **The radical.** An element ν of the ring R is a *radical-element*, if it belongs to at least one nilpotent ideal. Since every nilpotent l-ideal and every nilpotent r-ideal is contained in a nilpotent ideal,[7] the elements of nilpotent l-ideals and r-ideals are radical-elements. The sum of two nilpotent ideals is a nilpotent ideal;[8] the same holds for any finite number of nilpotent ideals. It follows readily that the set of all radical-elements forms an ideal N, the *radical* of R. It is easy to give examples of rings R whose radical N is not nilpotent. Hence we have to make a further assumption.

ASSUMPTION (A). *If Σ is a nonvacuous set of ideals $\mathfrak{a}$ which consist of nilpotent elements of R, then there exists at least one minimal ideal of Σ.*

[7] Cf. A. A. Albert, loc. cit., p. 22.

[8] Cf. A. A. Albert, loc. cit., p. 23.

We now prove this theorem.

THEOREM 1. *If the ring R satisfies this assumption* (A), *then its radical N is nilpotent.*

PROOF. (a) Let us first suppose that the ring R even satisfies the assumption (A) when the word "ideal" in it is replaced by the word "left-ideal."[9,10]

We have

$$N \supseteq N^2 \supseteq N^3 \supseteq \cdots . \tag{3}$$

Since all these ideals consist of nilpotent elements of R, there exists a minimal ideal $N^k = T$ of the set (3). If $T = 0$, we are finished. Assume $T \neq 0$. Then

$$T^2 = T. \tag{4}$$

Consider the set Σ of all l-ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a} \neq 0$. This set is not empty, since it contains $\mathfrak{a} = T$. Let $\mathfrak{a}$ be a minimal l-ideal of Σ. Since $T\mathfrak{a} \neq 0$, there exists an element α in $\mathfrak{a}$ such that $T\alpha \neq 0$. Then $T\alpha \subseteq \mathfrak{a} \subseteq T$ and $T(T\alpha) = T^2\alpha = T\alpha \neq 0$. Hence $T\alpha$ itself belongs to Σ. Since $\mathfrak{a}$ was minimal, we have

$$\mathfrak{a} = T\alpha. \tag{5}$$

In particular, the element α of $\mathfrak{a}$ belongs to $T\alpha$. We can find an element τ of T such that $\alpha = \tau\alpha$. This implies $\alpha = \tau\alpha = \tau^2\alpha = \tau^3\alpha = \cdots$. However, τ as an element of $T = N^k$ is nilpotent. Hence $\tau^l\alpha = 0$ for a suitable l, and we obtain $\alpha = 0$ which contradicts $T\alpha \neq 0$. This proves Theorem 1 under our present assumption.

(b) If we assume that R satisfies the assumption (A) in its original form, we have to replace the set Σ by the set Σ' of all ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a}T \neq 0$. Again, the ideal T belongs to the set. If $\mathfrak{a}$ is a minimal ideal of Σ', we can find an element α of $\mathfrak{a}$ such that $T\alpha T \neq 0$. Then $T\alpha T$ belongs to Σ', and the minimal property of $\mathfrak{a}$ gives

$$\mathfrak{a} = T\alpha T. \tag{6}$$

Consequently, the element α of $\mathfrak{a}$ belongs to $T\alpha T$. This means that there exist elements $\tau_1, \tau_2, \cdots, \tau_n, \tau_1', \tau_2', \cdots, \tau_n'$ in T such that

[9] For the proof of the theorem, it is not necessary to deal with this case separately. However, the proof becomes somewhat simpler when we make the stronger assumption. The minimum condition for l-ideals of R, implies this stronger assumption.

[10] Added July 5, 1942: The proof in (a) was found independently by Reinhold Baer.

$$\alpha = \sum_{i=1}^{n} \tau_i \alpha \tau_i' .$$

On replacing α on the right side by $\sum \tau_j \alpha \tau_j'$ and continuing in this manner, we obtain

$$(7) \quad \alpha = \sum_{i} \tau_i \alpha \tau_i' = \sum_{i,j} \tau_i \tau_j \alpha \tau_j' \tau_i' = \sum_{i,j,k} \tau_i \tau_j \tau_k \alpha \tau_k' \tau_j' \tau_i' = \cdots .$$

The radical element τ_i belongs to a nilpotent ideal $\mathfrak{n}_i$. Hence the sum $\mathfrak{q}$ of the n ideals $\mathfrak{n}_i$ is a nilpotent ideal containing all τ_i. If $\mathfrak{q}^r = 0$, then the rth of the sums in (7) vanishes since all products of r factors τ_i $(1 \leqq i \leqq n)$ will vanish. Hence $\alpha = 0$, which contradicts the condition $T\alpha T \neq 0$. This proves the theorem.

4. **Existence of idempotents.** For the last two sections, we make the following assumptions concerning the ring R:

(I) *The radical N of R is nilpotent.*

(II) *If Σ is a nonvacuous set of l-ideals $\mathfrak{a} \supseteq N$, there exists at least one minimal l-ideal of Σ.*

The condition (A), §3, implies the condition (I) as is shown by Theorem 1. If R satisfies the minimum condition for l-ideals, then certainly (A) and (I) hold, that is, (I) and (II) hold.

We say that an l-ideal is *regular*, if it is not nilpotent. An l-ideal $\mathfrak{a}$ is *primitive*, if $\mathfrak{a}$ is regular while every l-ideal $\mathfrak{b}$ with $\mathfrak{b} \subset \mathfrak{a}$ is nilpotent.

LEMMA 1. *Every regular l-ideal $\mathfrak{m}$ contains an element η with $\eta^2 \equiv \eta$, $\eta \not\equiv 0 \pmod{N}$.*

PROOF.[11] (a) Assume first that $\mathfrak{m} \supset N$. Using the assumption (II), we obtain an l-ideal $\mathfrak{a}$ with $\mathfrak{m} \supseteq \mathfrak{a} \supset N$ such that no l-ideal lies between $\mathfrak{a}$ and N. If $\mathfrak{a}\alpha \subseteq N$ for all α in $\mathfrak{a}$, we have $\mathfrak{a}^2 \subseteq N$ which would imply that $\mathfrak{a}^2$ is nilpotent. But then $\mathfrak{a}$ is nilpotent, that is, $\mathfrak{a} \subseteq N$. Hence for a suitable α in $\mathfrak{a}$, the l-ideal $\mathfrak{a}\alpha$ does not belong to N. Then $N \subset N + \mathfrak{a}\alpha \subseteq \mathfrak{a}$.[12] It follows that

$$\mathfrak{a} = N + \mathfrak{a}\alpha.$$

This implies that α can be written in the form $\alpha = \nu + \eta\alpha$ with ν in N and η in $\mathfrak{a}$. Then $\eta\alpha \equiv \alpha \pmod{N}$, and hence $\eta^2\alpha \equiv \eta\alpha$, $(\eta^2 - \eta)\alpha \equiv 0$

[11] If the minimum condition for l-ideals is satisfied in R, this proof can be simplified as follows: The l-ideal $\mathfrak{m}$ contains a primitive l-ideal $\mathfrak{a}$. As in the proof, we may choose an α in $\mathfrak{a}$ such that $\mathfrak{a}\alpha$ does not lie in N. Then $\mathfrak{a}\alpha = \mathfrak{a}$. This gives the existence of an η in $\mathfrak{a}$ for which $\eta\alpha = \eta$. As in the proof, we can conclude $\eta^2 \equiv \eta \not\equiv 0 \pmod{N}$.

[12] We use the $+$ sign, even if the sum of the l-ideals is not direct.

(mod N). The elements x of $\mathfrak{a}$ for which $x\alpha \equiv 0 \pmod{N}$ form an l-ideal $\mathfrak{b}$ with $N \subseteq \mathfrak{b} \subseteq \mathfrak{a}$. However, η does not lie in $\mathfrak{b}$, since $\eta\alpha \equiv 0 \pmod{N}$ would imply $\alpha \equiv 0 \pmod{N}$ and $\mathfrak{a}\alpha \subseteq N$. Hence $\mathfrak{b} \neq \mathfrak{a}$, that is, $\mathfrak{b} = N$. The element $\eta^2 - \eta$ lies in $\mathfrak{b}$, which gives $\eta^2 \equiv \eta \pmod{N}$. If we had $\eta \equiv 0 \pmod{N}$, then again $\alpha \equiv \eta\alpha \equiv 0 \pmod{N}$, which was impossible. Hence $\eta \not\equiv 0$, $\eta^2 \equiv \eta \pmod{N}$ and η lies in $\mathfrak{a}$.

(b) If $\mathfrak{m}$ does not contain N, set $\mathfrak{m}^* = \mathfrak{m} + N$. Then, as shown in (a), the l-ideal $\mathfrak{m}^*$ contains an element η^* with $\eta^{*2} \equiv \eta^* \not\equiv 0 \pmod{N}$. However, every η^* of $\mathfrak{m} + N$ is congruent to an element η of $\mathfrak{m}$, and this η will satisfy the conditions of Lemma 1.

LEMMA 2. *If r is a given positive integer, we may find a polynomial $f(x)$ with rational integral coefficients such that*

$$f(x) \equiv 0 \pmod{x^{r+1}}, \qquad f(x) \equiv 1 \pmod{(1-x)^r}. \tag{8}$$

PROOF. Expand the square bracket on the right side of $1 = 1^{2r} = [x + (1-x)]^{2r}$ according to the binomial theorem. If $f(x)$ is the sum of the terms containing x at least to the power x^{r+1}, then $f(x)$ satisfies the congruences (8).

THEOREM 2. *Every regular l-ideal $\mathfrak{m}$ contains an idempotent ϵ.*

PROOF. Construct η according to Lemma 1. Then $(\eta - \eta^2)^r = 0$ for some r. The element $\epsilon = f(\eta)$ is well defined, as $f(x)$ has no constant term. It follows from (8) that we have an equation $f(x)^2 - f(x) = (x - x^2)^r g(x)$ where $g(x)$ is a polynomial with rational integral coefficients such that $g(x)$ has no constant term. If we replace x by η, we obtain $\epsilon^2 - \epsilon = 0$. If we had $\epsilon = f(\eta) = 0$, we could multiply the second congruence (8) by x^{r+1} and replace x by η. This would give $0 = \eta^{r+1}$ which contradicts the congruences $\eta \equiv \eta^2 \equiv \eta^3 \equiv \cdots$, $\eta \not\equiv 0 \pmod{N}$. Hence ϵ is an idempotent belonging to $\mathfrak{a}$.

COROLLARY. *An element ν of R is a radical element, if it is properly nilpotent, that is, if $\alpha\nu$ is nilpotent for every α in R.*

PROOF. If ν belongs to the nilpotent ideal $\mathfrak{n}$, then $R\nu \subseteq \mathfrak{n} \subseteq N$, and all $\alpha\nu$ are nilpotent. If ν is properly nilpotent, then $\mathfrak{a} = R\nu$ cannot contain an idempotent. Hence $R\nu \subseteq N$. The set of all ν for which $R\nu \subseteq N$, forms an ideal $\mathfrak{n}$ which again cannot contain an idempotent. Hence $\mathfrak{n} \subseteq N$; in particular, ν belongs to N.

5. **Primitive l-ideals contained in regular l-ideals.**[13] We prove the following theorems.

[13] If the minimum condition for l-ideals is assumed, Theorem 4 becomes trivial.

THEOREM 3. *Let $\mathfrak{a}$ be an l-ideal with $\mathfrak{a} \supseteq N$, such that no l-ideal lies between $\mathfrak{a}$ and N. If ϵ is an idempotent belonging to $\mathfrak{a}$, then $R\epsilon$ is a primitive l-ideal contained in $\mathfrak{a}$.*

PROOF. Suppose $\mathfrak{b}$ is a regular l-ideal with $\mathfrak{b} \subset R\epsilon$. Then $\mathfrak{b}$ contains an idempotent ϵ' and we have

$$R\epsilon' \subset R\epsilon. \tag{9}$$

Since ϵ' belongs to $R\epsilon$, we have $\epsilon'\epsilon = \epsilon'$. Set $\xi = \epsilon - \epsilon\epsilon'$. Then $\xi\epsilon = \epsilon^2 - \epsilon\epsilon'\epsilon = \epsilon - \epsilon\epsilon' = \xi$, $\xi\epsilon' = 0$. Hence $\xi^2 = \xi\epsilon - \xi\epsilon\epsilon' = \xi$.

If $\xi \neq 0$, it is an idempotent contained in $R\epsilon$. Then $N \subseteq R\xi + N \subseteq \mathfrak{a}$. Since no l-ideal lies between $\mathfrak{a}$ and N and $R\xi$ contains $\xi^2 = \xi \not\equiv 0 \pmod{N}$, we have $R\xi + N = \mathfrak{a}$. This implies $\mathfrak{a}\epsilon' = R\xi\epsilon' + N\epsilon' = N\epsilon' \subseteq N$. However, $\mathfrak{a}\epsilon'$ contains $\epsilon'^2 = \epsilon'$ which does not lie in N; we have a contradiction.

Hence $\xi = 0$, that is, $\epsilon\epsilon' = \epsilon$. Then $R\epsilon'$ contains $\epsilon\epsilon' = \epsilon$, and $R\epsilon' \supseteq R\epsilon$. This contradicts (9), and the theorem is proved.

THEOREM 4. *Every regular l-ideal $\mathfrak{m}$ contains a primitive l-ideal.*

PROOF. Let $\mathfrak{a}$ be an l-ideal such that $N \subset \mathfrak{a} \subseteq \mathfrak{m} + N$ and that no l-ideal lies between N and $\mathfrak{a}$. Then $\mathfrak{a}$ contains an idempotent ϵ_0, and $\epsilon_0 = \eta + \nu$ with η in $\mathfrak{m}$ and ν in N. Hence $\eta^2 \equiv \epsilon_0^2 = \epsilon_0 \equiv \eta \pmod{N}$, $\eta \equiv \epsilon_0 \not\equiv 0 \pmod{N}$. Using Lemma 2 as in the proof of Theorem 2, we obtain an idempotent $\epsilon = f(\eta)$ which belongs to $\mathfrak{m}$. Then $R\epsilon \subseteq \mathfrak{m}$. Since $\eta = \epsilon_0 - \nu$ lies in $\mathfrak{a} + N = \mathfrak{a}$, the element $\epsilon = f(\eta)$ lies in $\mathfrak{a}$. Theorem 3 shows that $R\epsilon$ is primitive.

We can now prove this theorem.

THEOREM 5. *Every l-ideal $\mathfrak{m}$ is a direct sum of primitive l-ideals $R\epsilon_i$ and a nilpotent l-ideal $\mathfrak{n}$:*

$$\mathfrak{m} = R\epsilon_1 + R\epsilon_2 + \cdots + R\epsilon_n + \mathfrak{n}.^{14} \tag{10}$$

Here the ϵ_i can be taken as idempotents such that

$$\epsilon_i\epsilon_j = 0 \quad \text{for} \quad i \neq j, \qquad \epsilon_i^2 = \epsilon_i, \qquad \mathfrak{n}\epsilon_i = 0. \tag{11}$$

PROOF. Because of the assumption (II), §4, we may assume that the theorem is correct for all regular r-ideals $\mathfrak{m}'$ with $\mathfrak{m}' + N \subset \mathfrak{m} + N$. Let $R\epsilon_n$ be a primitive l-ideal contained in $\mathfrak{m}$, ϵ_n an idempotent, and apply the Peirce decomposition. Then $\mathfrak{m}$ is a direct sum

$$\mathfrak{m} = \mathfrak{m}' + R\epsilon_n \tag{12}$$

[14] If $\mathfrak{m}$ is nilpotent, the terms $R\epsilon_i$ are missing.

where $\mathfrak{m}'$ consists of those elements μ of $\mathfrak{m}$ for which $\mu\epsilon_n=0$. This implies $\mathfrak{m}'+N\subset\mathfrak{m}+N$, as $(\mathfrak{m}'+N)\epsilon_n=N\epsilon_n\subseteq N$ while $(\mathfrak{m}+N)\epsilon_n$ contains ϵ_n. Then Theorem 5 holds for $\mathfrak{m}'$. If $\mathfrak{m}'=R\epsilon_1+\cdots+R\epsilon_{n-1}+\mathfrak{n}$ is the corresponding representation, then (12) gives the representation (10) of $\mathfrak{m}$. However, we obtain the formula (11), $\epsilon_i\epsilon_j=0$, only for $i\neq n$. We must replace ϵ_i $(i=1, 2, \cdots, n-1)$ by $\epsilon_i-\epsilon_n\epsilon_i$ in order to have $\epsilon_n\epsilon_i=0$. As is easily seen, these new elements satisfy all the conditions.

THEOREM 6. *If Theorem* 5 *is applied to* $R=\mathfrak{m}$, *then* $\zeta=\epsilon_1+\epsilon_2+\cdots+\epsilon_n$ *is a* 1*-element* (mod N), *that is,* $\alpha\zeta\equiv\zeta\alpha\equiv\alpha$ (mod N) *for all* α *in* R. *If* R *has a* 1*-element* 1, *then* $\zeta=1$, *and in the representation* (10) *of* $\mathfrak{m}=R$ *no term* $\mathfrak{n}$ *appears.*

PROOF. If we represent an element μ of R in accordance with (10) for $\mathfrak{m}=R$ we obtain easily from (11) that $\mu\zeta\equiv\mu$ (mod N). For any α in R, we then have $\mu(\zeta\alpha-\alpha)\equiv0$ (mod N) for every μ in R. Consequently, $\zeta\alpha-\alpha$ is properly nilpotent, that is, $\zeta\alpha\equiv\alpha$ (mod N). If R contains a 1-element 1, then $\mu(1-\zeta)\equiv\mu-\mu\equiv0$ (mod N) which proves that $1-\zeta$ is properly nilpotent. Since $(1-\zeta)^2=1-\zeta-\zeta+\zeta^2=1-\zeta$, the element $1-\zeta$ is either 0 or an idempotent. The latter case is excluded, hence $\zeta=1$. Finally, $\mu=\mu1=\mu\zeta=\sum\mu\epsilon_i$ which shows that no term $\mathfrak{n}$ appears in this case.

Theorems 5 and 6 form the basis for the structure theory of rings, and for the theory of representations of rings.

UNIVERSITY OF TORONTO AND
INSTITUTE FOR ADVANCED STUDY

ON AN ELEMENTARY ANALOGUE OF THE RIEMANN-MANGOLDT FORMULA

AUREL WINTNER

Ramanujan's unsuccessful approach to the Prime Number Theorem, published only recently, is based on the power series

$$\sum_{n=1}^{\infty} \Lambda(n)x^n \equiv \sum_{p} \log p \sum_{k=1}^{\infty} x^{pk} \quad \text{and} \quad \sum_{n=1}^{\infty} p^n x^{p^n}, \qquad 0 < x < 1,$$

where p denotes a prime and p is 2 in the last series. In his discussion of Ramanujan's failure in case of the latter series, a series impracticable as $x \to 1$, Hardy gives for the function represented by the series another expansion, one exhibiting the critical "wobbles," as follows:[1]

$$\sum_{n=1}^{\infty} p^n \exp(-p^n s) = \{\ \}/\log p, \qquad p = 2, \tag{1}$$

where $\{\ \}$ is the expression

$$\{\ \} = \frac{1}{s} - \frac{1}{s} \sum_{k=-\infty}^{\infty}{}' \Gamma\left(\frac{1 + 2\pi k i}{\log p}\right) s^{-2\pi k i/\log p} - \log p \sum_{n=0}^{\infty} \frac{(-1)^n p^{n+1}}{p^{n+1} - 1} \frac{s^n}{n!}; \tag{2}$$

it being understood that $\sum_{k=-\infty}^{\infty}{}' = \sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}$ and $-\log x = s > 0$.

It will be seen later on that Hardy's result (2) contains two errors. However, the purpose of this note is not calculation of the corrections necessary, which are of a trivial nature, but the presentation of a short approach which seems to be of methodical and historical interest.

First, (2) is of the same type as the "explicit formula" of Riemann-Mangoldt[2] (the two sums representing the contributions of the "non-trivial" and "trivial" zeros, respectively). Correspondingly, Hardy's

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[1] G. H. Hardy, *Ramanujan*, Cambridge, 1940, chap. II, formulae (2.9.1) and (2.11.2).

[2] Cf., for example, A. E. Ingham, *The Distribution of Primes*, Cambridge Tracts, no. 30 (1932), chap. IV.

Relevant for the comparison is only the "Abelian" form (instead of the deeper "Cesàro" form) of the Riemann-Mangoldt formula; cf. G. H. Hardy and J. E. Littlewood, Acta Mathematica, vol. 41 (1918), pp. 119–196.

proof, omitted loc. cit.,[1] is based on a contour integration.[2] However, since the "non-trivial" zeros occurring in the part $\sum'$ of (2) are equidistant, the "wobbly" terms now become *purely* periodic, indicating the possibility of a trivial approach.[4]

It turns out that such an approach has been found already by Dedekind. What makes it of particular interest is the fact that Dedekind's purpose was precisely the explicit illustration of the illegitimacy of the argument to occur decades later in Ramanujan's attempted proof of the Prime Number Theorem. Inasmuch as Dedekind's result, published in a certainly conspicuous work,[5] has never[6] been quoted, it is possible that his indications were found to be hard to follow. All of this seems to warrant a detailed and straightforward deduction of the explicit formula.

To this end, put

$$f(s) = \sum_{n=-\infty}^{\infty} c^n \exp(-c^n s), \tag{3}$$

where c is any constant satisfying $c>1$. Then (3) obviously represents a regular function[7] for $s>0$. It is also seen that (3) remains unchanged if n is replaced by $n+1$ in the exponentials; so that[8] $sf(s)$ remains unaltered if s is replaced by cs. It follows that, if $\log c>0$ and $s>0$,

$$f(s)\log c = s^{-1}\sum_{k=-\infty}^{\infty}\Gamma(1+Ck\hat{\imath})s^{-Ck\hat{\imath}}, \qquad \text{where } C = 2\pi/\log c \tag{4}$$

(it being understood that $s^{\alpha\hat{\imath}}$ is meant to be defined in terms of the principal branch of $\log s$).

In fact, since $sf(s)=csf(cs)$, the function $e^u f(e^u)$ of the real variable $u=\log s$ has the period $\log c$. Since this function, being regular, is continuous and of bounded variation, the Fourier expansion

[1] G. H. Hardy, *Lectures on the Mathematical Work of Ramanujan*, The Institute for Advanced Study, 1936, p. 14.

[4] Cf. Ingham's comments on $\sum'\sin 2\pi nx/n$, loc. cit., p. 81.

[5] Supplement IX of Dirichlet's *Vorlesungen ueber Zahlentheorie* (pp. 385–386 of the 3rd edition (1879)).

The non-differentiable function of Weierstrass depends on a limiting case of Dedekind's series. In this connection, cf. G. H. Hardy and J. E. Littlewood, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 583–586.

[6] Cf. Hardy's paper of 1907, quoted loc. cit., Footnote 1, p. 47, §2.11.

[7] It will always be assumed that $s>0$, the transition from the half-line $s>0$ to the half-plane $R(s)>0$ being trivial.

[8] This remark of Dedekind corresponds to the more elaborate argument (loc. cit., Footnote 1, p. 39) which Hardy's paper of 1907 attributes to Wedderburn (cf. loc. cit., Footnote 6).

$$e^u f(e^u) = \sum_{k=-\infty}^{\infty} \alpha_k e^{-Ckiu},$$

where

$$\alpha_k = \frac{1}{\log c} \int_0^{\log c} e^{Ckiu} e^u f(e^u) du, \qquad C = \frac{2\pi}{\log c},$$

is valid for every $u = \log s$. Hence, in order to prove (4), it is sufficient to verify that the last integral is $\Gamma(1+Cki)$. But the uniformity of the convergence of the series (3) assures that this integral is

$$\sum_{n=-\infty}^{\infty} c^n \int_0^{\log c} e^{Ckiu} e^u \exp(-c^n e^u) du$$

or, since $\int_0^a F(u)du \equiv \int_{na}^{(n+1)a} F(u-na)du$ and $C = 2\pi/\log c$,

$$\sum_{n=-\infty}^{\infty} c^n \int_{n \log c}^{(n+1) \log c} e^{Ckiu} e^{u-n \log c} \exp(-c^n e^{u-n \log c}) du,$$

that is,

$$\sum_{n=-\infty}^{\infty} c^n \int_{n \log c}^{(n+1) \log c} e^{Ckiu} e^u c^{-n} \exp(-c^n e^u c^{-n}) du$$

or simply

$$\sum_{n=-\infty}^{\infty} \int_{n \log c}^{(n+1) \log c} e^{(1+Cki)u} \exp(-e^u) du \equiv \int_{-\infty}^{\infty} e^{(1+Cki)u} \exp(-e^u) du.$$

Since the last integral is reduced by $u = \log s$ to the integral defining $\Gamma(1+z)$ for $z = Cki$, the proof of the Dedekind expansion (4) is complete.

While the (real) series (3) converges only[9] for $s > 0$, its "non-principal part," that is, the series

$$g(s) = \sum_{m=1}^{\infty} c^{-m} \exp(-c^{-m}s), \tag{5}$$

converges for every s, since $c > 1$. Furthermore, the function (5) has the Taylor series

$$g(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+1} - 1} \frac{s^n}{n!} \tag{6}$$

[9] Incidentally, the function (3) of the complex variable s, where $R(s) > 0$, has the line $R(s) = 0$ as natural boundary. In fact, (3) and (5) show that $f(s) - g(s)$ is a lacunary Dirichlet series for every $c > 1$. On the other hand, (5) or (6) implies that $g(s)$ is an entire function.

for every s. In fact, substitution of the geometric progression

$$(c^{n+1} - 1)^{-1} = \sum_{m=1}^{\infty} c^{-(n+1)m}, \qquad c > 1,$$

transforms (6) into an iterated series which, being absolutely convergent, can be rearranged into

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n c^{-(n+1)m} s^n / n!.$$

But this may be written in the form

$$\sum_{m=1}^{\infty} c^{-m} \sum_{n=0}^{\infty} (-c^{-m}s)^n / n!,$$

which is identical with (5).

According to (3) and (5),

$$\sum_{n=0}^{\infty} c^n \exp(-c^n s) = f(s) - g(s). \tag{7}$$

If (4) and (6) are substituted into (7), the product of the Dirichlet series (7) into s appears as decomposed into a Fourier series in $\log s$ and an entire power series in s.

If the power series of $\exp(-s)$ is added to (5) and (6), then (7) appears in the form

$$\sum_{n=1}^{\infty} c^n \exp(-c^n s) = f(s) - \sum_{n=0}^{\infty} \frac{(-1)^n c^{n+1}}{c^{n+1} - 1} \frac{s^n}{n!}. \tag{8}$$

Hence, (4) implies Hardy's explicit formula (2) for (1), if $-\sum'$ is corrected to $+\sum'$ and

$$\Gamma\left(\frac{1 + 2\pi k i}{\log p}\right) \quad \text{to} \quad \Gamma\left(1 + \frac{2\pi k i}{\log p}\right).$$

THE JOHNS HOPKINS UNIVERSITY

A NOTE ON THE LOWER SEMI-CONTINUITY OF DOUBLE INTEGRALS IN THE PARAMETRIC FORM

WILLIAM SCOTT

Consider a double integral problem of the calculus of variations in parametric form, that is, a problem involving an integral of the form

$$\int\int_B f(x, X)du$$

where x stands for the three coordinate functions, $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$, and X stands for their three Jacobians.

Various theorems of the following general type have been proved by McShane,[1] Caccioppoli,[2] Cimmino,[3] and Radó:[4]

If (i) the triples of functions $x_n^i(u^1, u^2)$, $i=1, 2$; $n=0, 1, 2, \cdots$, satisfy certain conditions; (ii) the surfaces defined by $x^i=x_n^i(u^1, u^2)$, $i=1, 2$; $n=1, 2, \cdots$, converge in some prescribed sense to the surface defined by $x_0^i=x_0^i(u^1, u^2)$, $i=1, 2$; (iii) $f(x, X)$ satisfies certain conditions; then

$$\liminf_n \int\int_{B_n} f(x_n(u), X_n(u))du \geqq \int\int_{B_0} f(x_0(u), X_0(u))du.$$

The conditions on $f(x, X)$ usually contain a condition on its Weierstrass E-function

$$E(x, X, \overline{X}) = f(x, \overline{X}) - \overline{X}^\alpha f_\alpha(x, X),$$

the summation convention being used for convenience in writing.

McShane[1] requires either $f>0$, $E\geqq 0$, or $f\geqq 0$, $E>0$. Caccioppoli[2] makes the same requirements. Cimmino[3] requires only $f\geqq 0$, $E\geqq 0$, but works with smooth surfaces.

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[1] McShane, *On the semi-continuity of double integrals in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 460–484.

[2] Caccioppoli, *Gli integrali doppi di forma parametrica nel calcolo delle variazioni*, Arri del Reale Istituto Veneto di Scienze, Lettere ed Arti, Anno Accademico, 1933-1934, vol. 93, Part 2.

[3] Cimmino, *Sulle condizione necessarie e sufficienti per la semi-continuita degle integrali doppi di forma parametrica*, Annali di Mathematica Pura ed Applicata, vol. 15 (1936), pp. 159–173. The author wishes to thank Professor McShane for calling his attention to this paper.

[4] Radó, *On the semi-continuity of double integrals in the parametric form*, Transactions of this Society, vol. 51 (1942), pp. 336–361. The author wishes to thank Professor Radó for access to the manuscript of this paper before it was published.

Radó[4] proves the semi-continuity theorems under the same restrictions on $f(x, X)$ as McShane and Caccioppoli, but for a more general class of surfaces; namely, the class of surfaces that admit of a parametric representation for which the Lebesgue area is given by the usual integral formula.

The question arises as to whether we cannot prove the semi-continuity theorem for the class of surfaces which Radó considers with the requirements that $f \geqq 0$, $E \geqq 0$. We shall first prove a lemma on the Weierstrass E-function, then prove a stronger form of Lemma 1.5 in Radó.[4] The remainder of the proof as given by Radó will then prove the semi-continuity theorem with $f \geqq 0$, $E \geqq 0$ if the revised form of Lemma 1.5 is used.

We shall use the notations and definitions in Radó[4] throughout the remainder of the paper.

LEMMA 1. *If f is admissible and x_0, X_0 are such that*

(1) $f(x_0, X_0) > 0$,

(2) $f(x_0, X) \geqq 0$,

(3) $E(x_0, X_0, X) \geqq 0$,

for every X, then there exists a $\delta > 0$ such that if $\overline{X}^a f_a(x_0, X_0) > 0$ then $f(x_0, \overline{X}) > \delta \|\overline{X}\|$.

PROOF. Since

$$X_0^a f_a(x_0, X_0) = f(x_0, X_0) > 0$$

at least one of $f_1(x_0, X_0), f_2(x_0, X_0), f_3(x_0, X_0)$ is different from 0. Hence the set of points $\overline{X}$ satisfying $\overline{X}^a f_a(x_0, X_0) > 0$, $\|\overline{X}\| = 1$, is an open hemisphere H^0. Since $E(x_0, X_0, \overline{X}) \geqq 0$ we have $f(x_0, \overline{X}) > 0$ for every $\overline{X} \in H^0$. If we deny the lemma, then there exists a sequence $\{\overline{X}_n\}$, $\overline{X}_n \in H^0$ such that $f(x_0, \overline{X}_n) \to 0$. Hence there exists a point $\overline{X}_0 \in H - H^0$ such that $f(x_0, \overline{X}_0) = 0$.

Let $H - H^0 = C$, and let the plane containing C be P. Let $d(\overline{X}, P)$ denote the distance of $\overline{X}$ from P. Then if $\overline{X} \in H$ we have

$$(*) \qquad \overline{X}^a f_a(x_0, X_0) = d(\overline{X}, P) \left(\sum_{i=1}^{3} f_i^2(x_0, X_0) \right)^{1/2}.$$

Let C' be the great circle with center at $X = 0$, unit radius, passing through $\overline{X}_0$, and perpendicular to C. Let the direction cosines of the tangent to C' at $\overline{X}_0$ toward H^0 be $\cos \phi_1$, $\cos \phi_2$, $\cos \phi_3$.

Let $\partial f(x_0, \overline{X}_0)/\partial \nu$ be the directional derivative of f along this tangent. Then we have

$$\frac{\partial f(x_0, \overline{X}_0)}{\partial \nu} = f_a(x_0, \overline{X}_0) \cos \phi_a.$$

On the one hand, we have from (*)

$$\frac{\partial f(x_0, \overline{X}_0)}{\partial v} \geqq \left(\sum_{i=1}^{3} f_i^2(x_0, X_0) \right)^{1/2} > 0,$$

and on the other hand, since $f(x_0, \overline{X}_0) = 0$, $f(x_0, X) \geqq 0$ we have

$$\frac{\partial f(x_0, \overline{X}_0)}{\partial v} = 0.$$

This is a contradiction. Hence the lemma is true.

LEMMA 2. *Let there be given an admissible f and a set of six constants $(x_0^1, x_0^2, x_0^3, X_0^1, X_0^2, X_0^3) = (x_0, X_0)$ such that*

(a) $f(x_0, X_0) > 0$,

(b) $f(x_0, X) \geqq 0$,

(c) $E(x_0, X_0, X) \geqq 0$,

for all X. Then there exist two positive constants δ_1, δ_2 such that the following holds. If η, $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, $\overline{X}^1$, $\overline{X}^2$, $\overline{X}^3$ satisfy the conditions

(α) $0 < \eta < \delta_1$,

(β) $\overline{X}^\alpha f_\alpha(x_0, X_0) > 0$,

(γ) $|f(\bar{x}, \overline{X}) - f(x_0, \overline{X})| \leqq \eta \|\overline{X}\|$,

then

$$f(\bar{x}, \overline{X}) - f(x_0, \overline{X}) + E(x_0, X_0, \overline{X}) \geqq -\eta\delta_2 \overline{X}^\alpha f_\alpha(x_0, X_0).$$

PROOF. The conditions of Lemma 1 are satisfied. Hence there exists a positive constant δ as described there. Let $\delta_1 = \delta/2$, $\delta_2 = 2/\delta$.

Case 1. $\overline{X}^\alpha f_\alpha(x_0, X_0) \geqq (\delta/2)\|\overline{X}\|$. Then

$$\begin{aligned} f(\bar{x}, \overline{X}) - f(x_0, \overline{X}) + E(x_0, X_0, \overline{X}) &\geqq -\eta\|\overline{X}\| \\ &\geqq -\eta\delta_2 \overline{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Case 2. $\overline{X}^\alpha f_\alpha(x_0, X_0) < (\delta/2)\|\overline{X}\|$. Then

$$\begin{aligned} &f(\bar{x}, \overline{X}) - f(x_0, \overline{X}) + E(x_0, X_0, \overline{X}) \\ &\quad = f(\bar{x}, \overline{X}) - \overline{X}^\alpha f_\alpha(x_0, X_0) \\ &\quad \geqq f(x_0, \overline{X}) - \eta\|\overline{X}\| - \overline{X}^\alpha f_\alpha(x_0, X_0) \\ &\quad \geqq 0 \\ &\quad \geqq -\eta\delta_2 \overline{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Therefore the lemma is true.

Let K_2 be the class of oriented surfaces whose Lebesgue area is given by the usual integral formula. Let ${}_\sigma S_0$, ${}_\sigma S_n$ be oriented surfaces in the class K_2 such that:

(I) ${}_nS_n \to {}_nS_0$.

(II) If $T_0: x_0^i = x_0^i(u^1, u^2)$, $(u^1, u^2) \in B_0$, $i = 1, 2, 3$ is a representation of ${}_nS_0$ then there exists, in x-space, a closed bounded set A such that $\sum(T_0, B_0) \subset A^0$, and $f(x, X) \geqq 0$ for $x \in A$ and for every vector X.

(III) For a. e. point $u \in B_0^0$ such that $X_0(u)$ exists and is different from 0 we have $E(x_0(u), X_0(u), \overline{X}) \geqq 0$ for every vector $\overline{X} \neq 0$.

As we mentioned above the proof given by Radó will now lead to the following result.

THEOREM. *If f is admissible and if conditions* (I), (II), *and* (III) are *satisfied then*

$$\liminf_n \iint_{B_n} f(x_n(u), X_n(u))du \geqq \iint_{B_0} f(x_0(u), X_0(u))du.$$

This theorem is final in a definite sense. We cannot enlarge the class of surfaces if we use ordinary Jacobians. Counterexamples have been given which show the necessity of the conditions $E \geqq 0$, $f \geqq 0$. Moreover if f is only assumed to be continuous, convex, and non-negative, then we do not necessarily have lower semi-continuity. For, let

$$f(x, X) = \left| X^3 + g(x^1, x^3)X^1 \right|$$

where

$$g(x^1, x^3) = \frac{\cos 16x^1}{2} \qquad \text{if } x^3 > 5/8,$$

$$g(x^1, x^3) = \frac{\cos n^4 x^1}{n} \qquad \text{if } 1/n - 1/n^3 \leqq x^3 \leqq 1/n + 1/n^3,\ n \geqq 2,$$

$$g(x^1, x^3) = \frac{\cos (n+1)^4 x^1}{n+1} + \frac{x^3 - (1/(n+1) + 1/(n+1)^3)}{(1/n - 1/n^3) - (1/(n+1) + 1/(n+1)^3)} \cdot \left[\frac{\cos n^4 x^1}{n} - \frac{\cos (n+1)^4 x^1}{n+1}\right]$$

$$\text{if } 1/(n+1) + 1/(n+1)^3 < x^3 < 1/n - 1/n^3,$$

$$g(x^1, x^3) = 0 \qquad \text{if } x^3 \leqq 0.$$

Then f satisfies all the conditions stated. Further let

$$x_0^1 = u^1, \qquad x_n^1 = u^1, \qquad 0 \leqq u^1 \leqq \pi,$$

$$x_0^2 = u^2, \qquad x_n^2 = u^2, \qquad 0 \leqq u^2 \leqq 1,$$

$$x_0^3 = 0, \qquad x_n^3 = \frac{1}{n} + \frac{\sin n^4 u^1}{n^3}.$$

Then we have

$$\liminf_n \iint_{B_n} f(x_n, X_n)du = \liminf_n \int_0^\pi \int_0^1 |1 - \cos^2 n^4 u^3| \, du^2 du^1$$

$$= \frac{\pi}{2} < \pi = \int_0^\pi \int_0^1 du^2 du^1$$

$$= \iint_{B_0} f(x_0, X_0)du.$$

THE OHIO STATE UNIVERSITY

A NON-INVOLUTORIAL SPACE TRANSFORMATION ASSOCIATED WITH A $Q_{1,n}$ CONGRUENCE

M. L. VEST

1. **Introduction.** The involutorial transformation associated with the congruence of lines meeting a curve of order m and an $(m-1)$-fold secant has been studied by DePaolis,[1] and Vogt[2] has studied the non-involutorial transformations for a linear congruence and bundle of lines. Cunningham[3] has recently studied some non-involutorial transformations associated with a $Q_{1,2}$ congruence. In the present paper a non-involutorial transformation associated with the congruence of lines on a plane curve of order n having an $(n-1)$-point and a secant through that point is considered. The bundle of lines through the multiple point is not considered as belonging to the congruence. The tangents to the curve at the point are considered to be distinct.

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[1] DePaolis, *Alcuni particolari transformazioni involutori dello spazio*, Rendiconti dell'Academia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 745–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlandische Kultur, class 84, 1906, pp. 8–16.

[3] Cunningham, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence*, this Bulletin, vol. 47 (1941), pp. 309–312.

Given the plane n-ic r, a line s meeting r at an $(n-1)$-point A, and two projective pencils of surfaces $|F_m|: s^{m-1}g_{2m-1}$ and $|F'_{m'}|: s^{m'-1}g_{2m'-1}$. Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t through $P(y)$, s, r meets the associated F' of $|F'|$ in one residual point $P'(x)$ the image of $P(y)$ under the transformation thus defined. The residual base curves of $|F|$ and $|F'|$, other than s, have been denoted by g and g', respectively. Through a point $O_{g'}$ on g' there is a unique line t' of the congruence, this line lying upon one surface of $|F'|$. The associated surface of $|F|$ meets t' in a point $\bar{P}$ which generates a curve $\bar{g}$. Similarly, beginning with a point O_g on g, a point $\bar{P}'$ generating a curve $\bar{g}'$ is found. It will be shown that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of the transformation, and that the point A is a fundamental point of the second kind.

2. **Equations of the transformation.** Let us take the equations of r and s, respectively, as

$$x_3[cx_1x_2] - [dx_1x_2] = 0, \qquad x_4 = 0, \qquad x_1 = x_2 = 0 \tag{1}$$

where

$$[cx_1x_2] = \sum_{i=0}^{n-1} c_{i,n-i-1}x_1^i x_2^{n-i-1}, \quad [dx_1x_2] = \sum_{j=0}^{n} d_{j,n-j}x_1^j x_2^{n-j}, \tag{2}$$

and the pencils of surfaces as

$$|F_m| \equiv U - uV = 0, \qquad |F'_{m'}| \equiv U' - uV' = 0 \tag{3}$$

where

$$U = (ax)\{ex_1x_2\} - (\alpha x)\{fx_1x_2\}, \qquad V = (bx)\{gx_1x_2\} - (\beta x)\{hx_1x_2\},$$
$$U' = (a'x)\{e'x_1x_2\} - (\alpha'x)\{f'x_1x_2\}, \quad V' = (b'x)\{g'x_1x_2\} - (\beta'x)\{h'x_1x_2\};$$

$$\{ex_1x_2\} = \sum_{p=0}^{m-1} e_{p,m-p-1}x_1^p x_2^{m-p-1}, \quad \{e'x_1x_2\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}x_1^p x_2^{m'-p-1} \tag{4}$$

$$(ax) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

and so on.

Through a generic point $P(y)$ there is an F of $|F|$ with parameter $u = U(y)/V(y)$, and to this corresponds the F' of $|F'|$ having equation

$$U'(x)V - V'(x)U = 0. \tag{5}$$

The unique transversal t through P, r, s meets (5) in the point P' having coordinates

$$(6)\qquad T^{-1}_{m+m'+n}:\begin{vmatrix}\sigma x_1 = Ry_1 + Ky_1[cy_1y_2] = y_1S,\\ \sigma x_2 = Ry_2 + Ky_2[cy_1y_2] = y_2S,\\ \sigma x_3 = Ry_3 + K[dy_1y_2],\\ \sigma x_4 = Ry_4,\end{vmatrix}$$

where

$$(7)\qquad \begin{aligned} K_{m+m'} &= U'V - V'U, \qquad R_{m+m'+n-1} = UW' - VZ',\\ W'_{m'+n-1} &= (b'z)\{g'y_1y_2\} - (\beta'z)\{h'y_1y_2\},\\ Z'_{m'+n-1} &= (a'z)\{e'y_1y_2\} - (\alpha'z)\{f'y_1y_2\},\\ (a'z) &= (a'_1y_1 - a'_2y_2)[cy_1y_2] + a'_3[dy_1y_2], \end{aligned}$$

and so on. Equations (6) are those of the inverse transformation.

In a similar manner the equations of the direct transformation are found to be

$$(8)\qquad T_{m+m'+n}:\begin{vmatrix}\tau y_1 = R'x_1 + K'x_1[cx_1x_2] = x_1S',\\ \tau y_2 = R'x_2 + K'x_2[cx_1x_2] = x_2S',\\ \tau y_3 = R'x_3 + K'[dx_1x_2],\\ \tau y_4 = R'x_4,\end{vmatrix}$$

where

$$(9)\qquad \begin{aligned} K'_{m+m'} &= UV' - U'V = -K, \quad R'_{m+m'+n-1} = U'W - V'Z,\\ W_{m+n-1} &= (bz)\{gx_1x_2\} - (\beta z)\{hx_1x_2\},\\ Z_{m+n-1} &= (az)\{ex_1x_2\} - (\alpha z)\{fx_1x_2\}. \end{aligned}$$

3. **Images of fundamental curves and elements.** The transformations T^{-1} and T applied to an F' and F of $|F'|$ and $|F|$, respectively, give $U'\sim(T^{-1})US^{m'-1}G$, $U\sim(T)U'S'^{m-1}G'$ where

$$(10)\qquad \begin{aligned} G_{2m'+n-1} &= W'U' - V'Z',\\ G'_{2m+n-1} &= WU - VZ,\\ S_{m+m'+n-1} &= UN' - VM',\\ S'_{m+m'+n-1} &= U'N - V'M,\\ M'_{m'+n-1} &= (a'w)\{e'y_1y_2\} - (\alpha'w)\{f'y_1y_2\},\\ M_{m+n-1} &= (aw)\{ex_1x_2\} - (\alpha w)\{fx_1x_2\},\\ N'_{m'+n-1} &= (b'w)\{g'y_1y_2\} - (\beta'w)\{h'y_1y_2\},\\ N_{m+n-1} &= (bw)\{gx_1x_2\} - (\beta w)\{hx_1x_2\},\\ (a'w) &= a'_3[dy_1y_2] - (a'_3y_3 + a'_4y_4)[cy_1y_2]. \end{aligned}$$

Here U and U' are the corresponding surfaces of $|F|$ and $|F'|$ and $g'\sim(T^{-1})G$, $s'\sim(T^{-1})S$, $g\sim(T)G'$, $s\sim(T)S'$.

Similarly

$$
\begin{aligned}
&K'\sim(T^{-1})\,K'S^{m+m'-2}GG', && K\sim(T)\,KS'^{m+m'-2}GG',\\
&K'\sim(T)\,K'S'^{m+m'-2}GG', && K\sim(T^{-1})\,KS^{m+m'-2}GG',\\
&G'\sim(T^{-1})\,RS^{2m+n-2}G', && G\sim(T)\,R'S'^{2m'+n-2}G,\\
&G'\sim(T)\,R'S'^{2m+n-2}G', && G\sim(T^{-1})\,RS^{2m'+n-2}G,\\
&R'\sim(T^{-1})\,S^{m+m'+n-2}GG', && R\sim(T)\,S'^{m+m'+n-2}GG',\\
&R'\sim(T)\,S'^{m+m'+n-2}[R'^2+K'(W'Z-WZ')], &&\\
&R\sim(T^{-1})\,S^{m+m'+n-2}[R^2+K(W'Z-WZ')], &&\\
&S'\sim(T^{-1})\,RS^{m+m'+n-3}GG', && S\sim(T)\,R'S'^{m+m'+n-3}GG',\\
&S'\sim(T)\,S'^{m+m'+n-3}\{R'^2S'+R'GG'+S'[GG'+K'(W'Z-WZ')]\}, &&\\
&S\sim(T^{-1})\,S^{m+m'+n-3}\{R^2S+RGG'+S[GG'+K(W'Z-WZ')]\}. &&
\end{aligned}
\tag{11}
$$

Through a point O_r on r there is a pencil of transversals through s. O_r determines an F' and the associated F cuts the pencil in s and a line l. The line l generates the ruled surface R, the image of r.

From a point O_s on s there is an n-ic cone of transversals to r, to each line of which corresponds one F of $|F|$ cutting that line in one point. The locus of all such points is a curve k which generates the surface S, the image of s. The order of k, determined by the intersection of S and a homaloidal surface, is $m+m'+n-2$.

Through a point $O_{g'}$ on g' there is a unique line t of the congruence, but every F of $|F|$ passes through $O_{g'}$, hence $O_{g'}\sim(T^{-1})t$. The ruled surface G generated by t is the image of g' under T. Furthermore, every point P' of the line determines the same F' and t meets the associated F in one point $\bar{P}$ so that $\bar{P}\sim(T)t$. The locus of points $\bar{P}$ is the curve $\bar{g}$ and $\bar{g}\sim(T)G$. The order of $\bar{g}$, determined by the intersection of two homaloidal surfaces, is $m+3m'+2n-3$. In a similar manner we find a curve $\bar{g}'$, of order $3m+m'+2n-3$, such that

$$\bar{g}'\sim(T^{-1})G'.$$

The multiple point A is a fundamental point of the second kind and has as an image $n-1$ lines $c_{1,i}$, $i=1,\cdots,n-1$, other than s lying one in each of the $n-1$ planes determined by s and the tangent lines to r at A.

We can now write the following correspondences:

$$
\begin{aligned}
&r \sim (T)\ R' : s^{m+m'+n-2} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&r \sim (T^{-1})\ R : s^{m+m'+n-2} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&s \sim (T)\ S' : r s^{m+m'+n-2} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&s \sim (T^{-1})\ S : r s^{m+m'+n-2} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&g \sim (T)\ G' : r s^{2m+n-2} g \bar{g}', \qquad g' \sim (T^{-1})\ G : r s^{2m'+n-2} g' \bar{g},\\
&\bar{g}' \sim (T)\ G' : r s^{2m+n-2} g \bar{g}', \qquad \bar{g} \sim (T)\ G : r s^{2m'+n-2} g' \bar{g}.
\end{aligned} \tag{12}
$$

4. **Invariant and homaloidal surfaces.** The eliminant of the parameter from $|F|$ and $|F'|$ is the pointwise invariant surface K. The plane $x_4 = 0$ and the planes determined by s and the tangent lines to r at A are also invariant, but not pointwise invariant.

Generic planes subjected to the transformations give

$$
\begin{aligned}
\pi' \equiv (A'x) &\sim (T^{-1})\ R(A'y) + K(A'z) = \phi_{m+m'+n},\\
\pi \equiv (Ay) &\sim (T)\ R'(Ax) + K'(Az) = \phi'_{m+m'+n},
\end{aligned}
$$

where the ϕ's are homaloidal surfaces of the transformations. Further,

$$
\phi \sim (T)(A'x) R' S'^{m+m'+n-2} G G', \qquad \phi' \sim (T^{-1})\ (Ay) R S^{m+m'+n-2} G G',
$$

hence the homaloidal webs are

$$
\infty^3 \left| \phi \right| : r s^{m+m'+n-2} g \bar{g}, \qquad \infty^3 \left| \phi' \right| : r s^{m+m'+n-2} g' \bar{g}'.
$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$
\begin{aligned}
H' &\equiv [\phi' \phi'] : r s^{m^2+m'^2+n^2+2mm'+2m'n-4m-4m'-4n+4} g' \bar{g}' k_{m+m'+n}\\
H &\equiv [\phi\phi] : r s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4} g \bar{g} k_{m+m'+n}.
\end{aligned}
$$

We now write the additional correspondences:

$$
\begin{aligned}
&\pi \sim (T)\ \phi' : r s^{m+m'+n-2} g' \bar{g}', \qquad \pi' \sim (T^{-1})\ \phi : r s^{m+m'+n-2} g \bar{g},\\
&K \sim (T)\ K' : s^{m+m'-2} g g' \bar{g} \bar{g}', \qquad K' \sim (T^{-1})\ K : s^{m+m'-2} g g' \bar{g} \bar{g}'.
\end{aligned} \tag{13}
$$

The jacobian of the transformation is $J = RGG'S$.

5. **Tangency along** s. The projectivity $y_1 = x_1$, $y_2 = x_2$, $y_3 = kx_3$, $y_4 = x_3 + x_4$ is applied to the fundamental surfaces of the transformation and an examination of the coefficients of the highest powers of x_3 shows K and S to have

$$
\begin{aligned}
D_{m+m'-2} \equiv\ & [(a'_3 k + a'_4)\{e' x_1 x_2\} - (\alpha'_3 k + \alpha'_4)\{f' x_1 x_2\}]\\
&\cdot [(b_3 k + b_4)\{g x_1 x_2\} - (\beta_3 k + \beta_4)\{h x_1 x_2\}]\\
&- [(b'_3 k + b'_4)\{g' x_1 x_2\} - (\beta'_3 k + \beta'_4)\{h' x_1 x_2\}]\\
&\cdot [(a_3 k + a_4)\{e x_1 x_2\} - (\alpha_3 k + \alpha_4)\{f x_1 x_2\}] = 0
\end{aligned}
$$

as common tangent surface along s. The same surface is tangent to K' and S' also along s.

6. **Intersection table.** Referring to (12), (13) and §5 we can now write the following intersection table:

$$[R'S']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6}g'\bar{g}'c_{1,1}\cdots c_{1,n-1}$$

$$[R'G']: s^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4}\bar{g}'$$

$$[R'\phi']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4}g'\bar{g}'l_{1,1}\cdots l_{1,n}$$

$$[R'K']: s^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4}g'\bar{g}'$$

$$[S'G']: rs^{2m^2+n^2+2mm'+3mn+m'n-8m-2m'-5n+6}\bar{g}'k_{1,1}\cdots k_{1,2m-2}$$

$$[S'\phi']: rs^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6}g'\bar{g}'k_{m+m'+n-2}$$

$$[S'K']: s^{m^2+m'^2+2mm'+mn+m'n-5m-5m'-2n+6+(m+m'-2)d}g'\bar{g}'$$

$$[G'\phi']: rs^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4}\bar{g}'j_{1,1}\cdots j_{1,2m-1}$$

$$[G'K']: s^{2m^2+mn+2mm'+m'n-6m-2m'-2n+4}g\bar{g}'$$

$$[\phi'K']: S^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4}g'\bar{g}'k_{m+m'}.$$

7. **The T_2 in a plane through s.** A plane $\pi \equiv x_1 = \sigma x_2$ cuts the surfaces of $|F_m|$ and $|F'_{m'}|$ in residual pencils of lines

$$|l| \equiv u - \mu v = 0, \qquad |l'| \equiv u' - \mu v' = 0$$

where $u = (ax)\{e\sigma\} - (\alpha x)\{f\sigma\}$, $u' = (a'x)\{e'\sigma\} - (\alpha' x)\{f'\sigma\}$, $v = (bx)\{g\sigma\} - (\beta x)\{h\sigma\}$, $v' = (b'x)\{g'\sigma\} - (\beta' x)\{h'\sigma\}$, $(ax) = (a_1\sigma + a_2)x_2 + a_3x_3 + a_4x_4$, $\{e\sigma\} = \sum_{p=0}^{m-1} e_{p,m-p-1}\sigma^p$, $\{e'\sigma\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}\sigma^p$, and so on. The n-ic r intersects π in one residual point P: $(\sigma[c\sigma], [c\sigma], [d\sigma], 0)$ where $[c\sigma] = \sum_{i=0}^{n-1} c_{i,n-i-1}\sigma^i$, $[d\sigma] = \sum_{j=0}^{n} d_{j,n-j}\sigma^j$, and the vertices of $|l|$ and $|l'|$ are designated as Γ and Γ'.

Through a generic point $P(y)$ of π passes one l of $|l|$ having parameter $\mu = u(y)/v(y)$ and to this corresponds the line $u'v - uv' = 0$ which is met by l in a point $P'(x)$, image of P under T. The T_2^{-1} in π is thus

$$x_2 = \rho y_2 + \kappa[c\sigma], \qquad x_3 = \rho y_3 + \kappa[d\sigma], \qquad x_4 = \rho y_4, \tag{14}$$

where $\rho_1 = uw' - vz'$, $\kappa_2 = u'v - v'u$, $z' = (a'q)\{e'\sigma\} - (\alpha' q)\{f'\sigma\}$, $w' = (b'q)\{g'\sigma\} - (\beta' q)\{h'\sigma\}$, $(a'q) = (a'_1\sigma + a'_2)[c\sigma] + a'_3[d\sigma]$, and so on.

The direct transformation T is

$$y_2 = \rho' x_2 + \kappa'[c\sigma], \qquad y_3 = \rho' x + \kappa'[d\sigma], \qquad y_4 = \rho' x_4 \tag{15}$$

where $\rho_1' = u'w - v'z$, $\kappa' = -\kappa$.

The conic $\kappa:\Gamma\bar{\Gamma}\Gamma'\bar{\Gamma}'$ is pointwise invariant under the transformation.

Through the point Γ there is one line γ' through P and every line of $|l'|$ corresponds to Γ, hence γ' is the image of Γ under T. Moreover, every point of γ' determines the same line of $|l'|$ which intersects γ' in a point $\bar{\Gamma}'$ whose image under T^{-1} is also γ'. Similarly, a γ of $|l|$ is the image of Γ' and $\bar{\Gamma}$ under T^{-1} and T, respectively.

The point P is the vertex of a pencil of transversals. Moreover, through P there passes one line of $|l|$, hence there corresponds one line of $|l'|$. This line is met in every point by a line of the pencil, hence is the image of P under T.

Thus the points $P, \Gamma, \bar{\Gamma}$ and $P, \Gamma', \bar{\Gamma}'$ are fundamental under T_2 and T_2^{-1}, respectively, so that we have $P\sim(T)\ \rho':\Gamma'\bar{\Gamma}'$, $\Gamma\sim(T)\ \gamma':P\Gamma\bar{\Gamma}'$, $\bar{\Gamma}\sim(T)\ \gamma:P\Gamma'\bar{\Gamma}$ and $P\sim(T)\ \rho:\Gamma\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})\ \gamma:P\Gamma'\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})\ \gamma':P\Gamma\bar{\Gamma}'$.

The homaloidal nets of T and T^{-1} are

$$\infty^2\left|f_2'\right|:P\Gamma'\bar{\Gamma}', \qquad \infty^2\left|f_2\right|:P\Gamma\bar{\Gamma}$$

while the jacobian of T_2 is $j=\rho'\gamma\gamma'$ and of T_2^{-1} is $j'=\rho\gamma\gamma'$.

As the plane π generates the pencil on s the T generates the space $T_{m+m'+n}$ whose equations may be obtained from (14) and (15) by replacing $u, u', v, v', w, w', z, z'$ and σ by $U, U', V, V', W, W', Z, Z'$ and x_1/x_2, respectively.

Since the point P is the section of r by π, r may be represented by $x_1=\sigma[c\sigma], x_2=[c\sigma], x_3=[d\sigma], x_4=0$.

8. **r having $(n-1)$-point with coincident tangents.** In case k of the tangents to r at A are coincident the transformation will be identical with the above except for the image of A, the correspondences involved then being

$$r\sim(T)\ R':s^{m+m'+n-2}\,g'\bar{g}'c_{1,1}'^{k}\,c_{1,k+1}'\cdots c_{1,n-1}',$$
$$s\sim(T)\ S':rs^{m+m'+n-3}\,g'\bar{g}'c_{1,1}'\,c_{1,k+1}'\cdots c_{1,n-1}',$$

and the intersection

$$[R'S']:s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6}\,g'\bar{g}'c_{1,1}'^{k}\,c_{1,k+1}'\cdots c_{1,n-1}'.$$

WEST VIRGINIA UNIVERSITY

HAUSDORFF MEANS INCLUDED BETWEEN $(C, 0)$ AND $(C, 1)$

HERBERT J. GREENBERG AND H. S. WALL

In this paper we show that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq \infty$ and $\phi(\infty) - \phi(0) = 1$, then the function $\alpha(z) = \int_0^\infty d\phi(u)/(1+zu)$ is a regular moment function; and we show that when $\phi(u)$ is further restricted to be monotone then the Hausdorff mean determined by $\alpha(z)$ is included between $(C, 0)$ and $(C, 1)$. Conditions under which this mean is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall[1] for the special case where $\phi(u) \equiv 1$ for $u \geqq 1$, $\phi(0) = 0$. In §1 we give an elementary development of the notion of Hausdorff summability; §2 contains a proof that $\alpha(z)$ is a regular moment function; §3 contains the above mentioned inclusion theorems; and §4 contains examples and a discussion of some transformations of moment functions which are suggested by the earlier developments.

1. **Hausdorff summability.** Let $A = (a_{ij})$ be any matrix in which $a_{ii} \neq 0$ and $a_{ij} = 0$ for $j > i$, $i, j = 0, 1, 2, \cdots$, and consider the system of equations

$$
\begin{aligned}
& a_{00}q_0 && = c_0(a_{00}p_0), \\
& a_{10}q_0 + a_{11}q_1 && = c_1(a_{10}p_0 + a_{11}p_1), \\
& a_{20}q_0 + a_{21}q_1 + a_{22}q_2 && = c_2(a_{20}p_0 + a_{21}p_1 + a_{22}p_2), \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned} \tag{1.1}
$$

These equations constitute a linear transformation of the sequence $\{p_n\}$ into the sequence $\{q_n\}$, the transformation depending upon the matrix A and the sequence $\{c_n\}$. If $\lim q_n = p$, we shall say that the sequence $\{p_n\}$ is $[A, c_n]$-*summable* to the limit p. A sequence $\{c_n\}$ such that $[A, c_n]$ sums every convergent sequence to its proper limit will be called A-*regular*. The following statements are almost obvious consequences of the above definitions:

(i) If $[A, c_n]$ transforms $\{p_n\}$ into $\{q_n\}$, and $[A, d_n]$ transforms $\{q_n\}$ into $\{r_n\}$, then $[A, c_n d_n]$ transforms $\{p_n\}$ into $\{r_n\}$.

(ii) If $\{c_n\}$, $\{d_n\}$ are A-regular, then $[c_n d_n]$ is A-regular.

(iii) If $[A, c_n]$ sums $\{p_n\}$ to the limit p, then $[A, kc_n]$ sums $\{p_n\}$ to the limit kp.

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[1] W. T. Scott and H. S. Wall, *Transformation of series and sequences*, Transactions of this Society, vol. 51 (1942), pp. 255–279.

(iv) If $\{c_n\}$, $\{d_n\}$ are A-regular, and $g+h=1$, then $\{gc_n+hd_n\}$ is A-regular.

(v) If $\{a_n\}$, $\{b_n\}$ are A-regular, and $b_n \neq 0$, $n=0, 1, 2, \cdots$, then $[A, a_n] \supset [A, b_n]$, that is, every sequence summable $[A, b_n]$ is summable $[A, a_n]$ to the same limit, if and only if $\{a_n/b_n\}$ is A-regular.

Hausdorff summability[2] is the particular $[A, c_n]$-summability obtained by taking $A=H=((-1)^j C_{ij})$ where $C_{ij}=i!/j!(i-j)!$, $0 \leqq j \leqq i$; $C_{ij}=0$, $j>i$. In this case the equations (1.1) may be written

$$\Delta^m q_0 = c_m \Delta^m p_0, \qquad m = 0, 1, 2, \cdots, \tag{1.2}$$

where $\Delta^i x_j = x_j - C_{i1}x_{j+1} + C_{i2}x_{j+2} - \cdots + (-1)^i C_{ii}x_{j+i}$; or, as may be readily shown:

$$q_m = \sum_{n=0}^{m} C_{mn} \Delta^{m-n} c_n \cdot p_n, \qquad m = 0, 1, 2, \cdots. \tag{1.3}$$

Inasmuch as we shall be dealing exclusively with Hausdorff summability from now on, we shall write *regular* instead of *H-regular*, whenever the occasion arises.

If $\{p_n\}$ is the sequence $1, 1, 1, \cdots$, then (1.2) becomes $\Delta^m q_0 = 0$, $m>0$, $\Delta^0 q_0 = q_0 = c_0$, or $q_m = c_0$, $m=0, 1, 2, \cdots$; and if $\{p_n\}$ is the sequence $1, 0, 0, \cdots$ then $\Delta^m q_0 = c_m$, $q_m = \Delta^m c_0$, $m=0, 1, 2, \cdots$. It follows that *necessary* conditions for $\{c_n\}$ to be regular are:

(a) $$c_0 = 1,$$

(b) $$\lim_{m=\infty} \Delta^m c_0 = 0.$$

Another necessary condition obtained by applying one of the well known conditions for the regularity of sequence transformations to (1.3) is:

(c) $$\sum_{n=0}^{m} C_{mn} \left| \Delta^{m-n} c_n \right| \leqq M, \qquad m = 0, 1, 2, \cdots,$$

where M is independent of m. These necessary conditions are also *sufficient*.[3]

Hausdorff showed that (c) holds if and only if $\{c_n\}$ is a *moment sequence*, that is,

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots, \phi(u) \in BV[0, 1]; \tag{c'}$$

[2] F. Hausdorff, *Summationsmethoden und Momentfolgen*, I and II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299.

[3] Hausdorff, loc. cit.

and that (a), (b) are equivalent, respectively, to the conditions:

(a′) $\phi(1) - \phi(0) = 1,$

(b′) $\phi(u)$ continuous at $u = 0.$

We recall that $(C, 0)$, or convergence, is the Hausdorff method $[H, c_n]$ where $c_n = 1$, $n = 0, 1, 2, \cdots$. Also, $(H, \alpha) = [H, (n+1)^{-\alpha}]$, $(E, r) = [H, r^n]$, $0 < r < 1$, are the Hölder method of order α and the Euler-Knopp method, respectively. The former is equivalent to $(C, \alpha) = [H, 1/C_{n+\alpha,n}]$, the Cesàro method of order α.

2. **Proof that $\{\alpha(n)\}$ is a regular sequence.** We define the function $\alpha(z)$ by the equation

$$(2.1) \qquad \alpha(z) = \int_0^\infty \frac{d\phi(u)}{1 + zu}, \qquad \phi(u) \in BV[0, \infty], \phi(\infty) - \phi(0) = 1,$$

and shall prove that the sequence $\{\alpha(n)\}$, $n = 0, 1, 2, \cdots$, is a moment sequence. To do this, it suffices to show that $\Delta^m\alpha(n) \geqq 0$, $m, n = 0, 1, 2, \cdots$, when $\phi(u)$ is monotone non-decreasing. But this is obvious from the formula

$$\Delta^m\alpha(n) = \int_0^\infty \frac{u^m m! d\phi(u)}{(1 + nu)(1 + [n+1]u) \cdots (1 + [n+m]u)}.$$

To show that $\{\alpha(n)\}$ is regular we must show that $\alpha(0) = 1$, and that $\Delta^m\alpha(0) \to 0$ as $m \to \infty$. That $\alpha(0) = 1$ follows from the hypothesis $\phi(\infty) - \phi(0) = 1$. Write $\Delta^m\alpha(0)$ as the sum of three integrals J_1, J_2, J_3, with the limits of integration 0 to 1, 1 to k, k to ∞, respectively, $k > 1$. Since the integrand

$$1 \Big/ \left[\left(1 + \frac{1}{u}\right)\left(1 + \frac{1}{2u}\right) \cdots \left(1 + \frac{1}{mu}\right)\right] < 1$$

it follows that if $\phi(u)$ is monotone then $0 \leqq J_3 \leqq \phi(\infty) - \phi(k) < \epsilon/3$, $\epsilon > 0$, if $k \geqq K_\epsilon$, ($K = K_\epsilon$ independent of m). If k is fixed, then

$$0 \leqq J_2 \leqq 1 \Big/ \left[\left(1 + \frac{1}{K}\right)\left(1 + \frac{1}{2K}\right) \cdots \left(1 + \frac{1}{mK}\right)\right] \leqq \frac{\epsilon}{3},$$

for all m sufficiently large, inasmuch as $\prod_{n=1}^{m}(1 + 1/nK)$ diverges to ∞. Also, $\lim_{m\to\infty} J_1 = 0$ inasmuch as the integrand tends monotonically to 0 for each u, $0 \leqq u \leqq 1$. Hence $\lim_{m\to\infty} \Delta^m\alpha(0) = 0$ if $\phi(u)$ is monotone. If $\phi(u) \in BV[0, \infty]$ it can be written as a linear combination of bounded monotone functions with coefficients ± 1, $\pm i$, and there-

fore $\lim_{m\to\infty} \Delta^m\alpha(0)=0$ in any case. We have completed the proof of the following theorem:

THEOREM 2.1. *The sequence* $\{\alpha(n)\}$, *where* $\alpha(z)$ *is given by* (2.1), *is a regular sequence.*

3. **An inclusion relation for** $[H, \alpha(n)]$ **when** $\phi(u)$ **is monotone.** In the integral defining $\alpha(z)$ replace z by $z+1$ and u by $v/(1-v)$, and we obtain:

$$\alpha(z+1)=\int_0^1 \frac{(1-v)d\phi_1(v)}{1+zv}, \qquad \phi_1(v)=\phi\left(\frac{v}{1-v}\right).$$

If $\phi(u)$ is monotone and $\phi(\infty)-\phi(0)=1$, then $\phi_1(v)$ is monotone, $0\leqq v\leqq 1$, and $\phi_1(1)-\phi(0)=1$. Consequently,[4] we have an expansion of the form

$$\alpha(z+1)=\frac{g_1}{1+}\ \frac{(1-g_1)g_2z}{1}\ +\ \frac{(1-g_2)g_3z}{1}\ +\cdots,$$

where $0\leqq g_n\leqq 1$, $n=1, 2, 3, \cdots$, and where the continued fraction is to be terminated with the first identically vanishing partial quotient in case some g_n is 0 or 1. Excepting in the trivial case $g_1=1$, $\alpha(z)\equiv 1$, we then have:

$$\frac{g_1}{\alpha(z+1)}=1+(1-g_1)z\left[\frac{g_2}{1+}\ \frac{(1-g_2)g_3z}{1}\ +\ \frac{(1-g_3)g_4z}{1}\ +\cdots\right]$$
$$=1+(1-g_1)z\alpha^*(z+1),$$

where

$$\alpha^*(z+1)=\int_0^1 \frac{(1-v)d\phi_1^*(v)}{1+zv},$$

a function of the same kind as $\alpha(z+1)$. That is, $\phi_1^*(v)$ is monotone; and $\phi_1^*(1)-\phi_1^*(0)=1$, since[5]

$$1=\phi_1(1)-\phi_1(0)=1-\left[1+\sum_{m=1}^{\infty}\frac{g_1g_2\cdots g_m}{(1-g_1)(1-g_2)\cdots(1-g_m)}\right]^{-1}$$

and $\phi_1^*(1)-\phi_1^*(0)$ is equal to the same expression with the subscripts of all the g's advanced by unity.

On replacing z by $z-1$ we therefore have the identity

[4] H. S. Wall, *Continued fractions and totally monotone sequences*, Transactions of this Society, vol. 48 (1940), pp. 165–184; p. 179 and p. 182.

[5] H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146–153; p. 147.

$$\frac{\alpha(1)}{\alpha(z)} = 1 + [1 - \alpha(1)](z - 1)\alpha^*(z), \qquad \alpha(1) < 1, \tag{3.1}$$

where

$$\alpha^*(z) = \int_0^\infty \frac{d\phi^*(u)}{1 + zu}, \qquad \phi^*(u) \text{ monotone, } \phi^*(\infty) - \phi^*(0) = 1$$

We are now prepared to prove the inclusion relation:

$$(C, 0) \subset [H, \alpha(n)] \subset (C, 1). \tag{3.2}$$

The left half is of course a restatement of the regularity of $[H, \alpha(n)]$, and (3.2) is obviously true if $\alpha(n) \equiv 1$. To prove the right half when $\alpha(1) < 1$ we write, by (3.1),

$$\frac{1}{(n+1)\alpha(n)} = \frac{1}{\alpha(1)}\left[\frac{1}{n+1} + [1-\alpha(1)]\alpha^*(n) - 2[1-\alpha(1)]\frac{\alpha^*(n)}{n+1}\right].$$

We have expressed the ratio $[1/(n+1)]:\alpha(n)$ as a linear combination of regular sequences where the constants of combination add up to unity. Hence by (iv), (v) of §1 it follows that $[H, \alpha(n)] \subset (C, 1)$.

If $\alpha(z)$ has a convergent Stieltjes[6] continued fraction expansion $1/a_1 + z/a_2 + z/a_3 + z/a_4 + \cdots$, where $a_1 = 1$, $a_n > 0$, $\sum a_n$ diverges, then there is the relation

$$1/\alpha(z) = 1 + za_2^{-1}\alpha^*(z), \tag{3.3}$$

where $\alpha^*(z) = a_2(1/a_2 + z/a_2 + z/a_3 + z/a_4 + \cdots)$, with the aid of which the inclusion $[H, \alpha(n)] \subset (C, 1)$ can be established. However, if $\sum a_n$ converges so that the continued fraction diverges, there is no assurance that a relation of the form (3.3) exists. The difficulty disappears as soon as one has (3.1).

We shall now proceed to obtain conditions under which one or the other of the inclusion symbols in (3.2) may be replaced by the equivalence symbol. In order to have $(C, 0) \approx [H, \alpha(n)]$ it is necessary and sufficient that $\{1/\alpha(n)\}$ be a regular sequence. In particular, $1/\alpha(n)$ must be bounded. Now $\lim_{n\to\infty} \alpha(n) = \phi(+0) - \phi(0)$, and therefore a *necessary* condition for $\{1/\alpha(n)\}$ to be regular is that $\phi(u)$ be discontinuous at $u = 0$. This condition is also *sufficient*. For, by (3.1) we may write, if $\alpha(1) < 1$:

$$\frac{1}{\alpha(n)} = \frac{1}{\alpha(1)}\left[1 - [1 - \alpha(1)]\alpha^*(n) + [1 - \alpha(1)]\int_0^\infty \frac{d\phi^*(u)}{(1/n) + u}\right].$$

[6] T. J. Stieltjes, *Oeuvres*, vol. 2, pp. 402–566.

Supposing $\phi(u)$ to be discontinuous at $u=0$, so that $1/\alpha(n)$ has a finite limit for $n=\infty$, this relation shows that $\int_0^\infty d\phi^*(u)/u<\infty$. Hence, if we put $\int_0^t d\phi^*(u)/u=\theta(t)$ we have:

$$\frac{1}{\alpha(n)}=\frac{1}{\alpha(1)}\left[1-[1-\alpha(1)]\alpha^*(u)+[1-\alpha(1)][\theta(\infty)-\theta(0)]-\int_0^\infty \frac{d\theta(u)}{1+nu}\right].$$

On applying (iv) of §1 we therefore see that $\{1/\alpha(n)\}$ is a regular sequence. In case $\alpha(1)=1$ so that $\alpha(n)\equiv 1$, the function $\phi(u)$ is discontinuous at $u=0$ and in this case $[H, \alpha(n)]\approx(C, 0)$.

In order to have $[H, \alpha(n)]\approx(C, 1)$ it is necessary and sufficient that $\{(n+1)\alpha(n)\}$ be a regular sequence. Since

$$(n+1)\alpha(n)=\int_0^\infty \frac{d\phi(u)}{1+nu}+\int_0^\infty \frac{d\phi(u)}{(1/n)+u},$$

we see as in the preceding that $\{(n+1)\alpha(n)\}$ is regular if and only if $\int_0^\infty d\phi(u)/u<\infty$. We have completed the proof of the following theorem:

THEOREM 3.1. *If* $\alpha(n)=\int_0^\infty d\phi(u)/(1+nu)$ *where* $\phi(u)$ *is monotone and* $\phi(\infty)-\phi(0)=1$, *then* $(C, 0)\subset[H, \alpha(n)]\subset(C, 1)$; *and* $(C, 0)\approx[H, \alpha(n)]$ *if and only if* $\phi(u)$ *is discontinuous at* $u=0$, *while* $[H, \alpha(n)]\approx(C, 1)$ *if and only if* $\int_0^\infty d\phi(u)/u<\infty$.

In case $\alpha(z)$ has a convergent Stieltjes continued fraction we have this theorem:

THEOREM 3.2. *If* $\alpha(z)=1/a_1+z/a_2+z/a_3+\cdots$, $a_1=1$, $a_n>0$, $\sum a_n$ *diverges, then* $[H, \alpha(n)]\approx(C, 0)$ *if and only if* $\sum a_{2n+1}$ *converges; and* $[H, \alpha(n)]\approx(C, 1)$ *if and only if* $\sum a_{2n}$ *converges.*

This follows from Theorem 3.1, together with the fact that $\phi(+0)-\phi(0)=1/\sum a_{2n+1}$,[7] and the fact that $\int_0^\infty d\phi(u)/u<\infty$ if and only if $\sum a_{2n}$ converges.[8]

In case the continued fraction for $\alpha(z)$ diverges, that is, $\sum a_n$ converges, then the odd approximants have one limit $\alpha_1(z)$ and the even approximants another limit $\alpha_2(z)$, one of which may equal $\alpha(z)$. In any case, $\alpha_1(z)$ and $\alpha_2(z)$ have integral representations of the form to which Theorem 2.1 applies. It is not difficult to show that $(C, 0)\approx[H, \alpha_1(n)]$, while $[H, \alpha_2(n)]\approx(C, 1)$. For[9] the integrals

[7] Stieltjes, loc. cit., p. 510.

[8] H. S. Wall, *On extended Stieltjes series*, Transactions of this Society, vol. 31 (1929), pp. 771–781; p. 774.

[9] Stieltjes, loc. cit., p. 403.

may be written as infinite series of the form $\sum_{i=0}^{\infty} M_i/(1+zu_i)$, $0 \leqq u_0 < u_1 < \cdots$, $M_i > 0$, where $u_0 = 0$ in the case of $\alpha_1(z)$ whereas $u_0 > 0$ in the case of $\alpha_2(z)$. Hence it follows that $\phi(u)$ is discontinuous at $u = 0$ in the case of $\alpha_1(z)$, while $\int_0^{\infty} d\phi(u)/u < \infty$ in the case of $\alpha_2(z)$.

4. **Some transformations of moment sequences.** If

$$\alpha(z) = \int_0^{\infty'} \frac{d\phi(u)}{1+zu}$$

where $\phi(u)$ is monotone and $\phi(\infty) - \phi(0) = 1$, then

$$\frac{1-\alpha(z)}{c_1 z} = \int_0^{\infty} \frac{(u d\phi(u)/c_1)}{1+zu}, \qquad c_1 = \int_0^{\infty} u d\phi(u),$$

defines a method of summation which is equivalent to $(C, 1)$ inasmuch as $\int_0^{\infty} (u d\phi(u)/c_1)/u < \infty$. It is noteworthy that there are other moment functions which are not of the form considered above and which have the property that the transformation

$$\alpha_1(z) = \frac{1-\alpha(z)}{c_1 z},$$

where c_1 is a normalizing factor such that $\alpha_1(0) = 1$, carries a regular moment function $\alpha(z)$ into another regular moment function $\alpha_1(z)$ such that $[H, \alpha_1(n)] \approx (C, 1)$. For instance, if $\alpha(z) = (1+z)^{-k}$, $k = 1, 2, 3, \cdots$, we find that

$$\alpha_1(z) = \frac{1}{k}\left(\frac{1}{(1+z)} + \frac{1}{(1+z)^2} + \cdots + \frac{1}{(1+z)^k}\right);$$

and it is not difficult to show that $[H, \alpha_1(n)] \approx (C, 1)$.

If $\alpha(z) = r^z$, $0 < r < 1$, so that $[H, \alpha(n)]$ is Euler-Knopp summability (E, r), then

$$\alpha_1(z) = \frac{1-r^z}{z \log(1/r)} = \int_0^1 u^z d\phi_1(u)$$

where

$$\phi_1(u) = \begin{cases} 0, & 0 \leqq u \leqq r, \\ 1 - (\log u/\log r), & r < u \leqq 1. \end{cases}$$

Since $\phi_1(u)$ is obviously a regular mass function it follows that $\alpha_1(z)$ is a regular moment function. *Although* $(C, 1)$ *and* (E, r) *are not comparable methods, nevertheless* $[H, \alpha_1(n)] \supset (C, 1)$. For,

$$\alpha_1(z) : (1+z)^{-1} = \int_0^1 u^z d\phi_2(u),$$

where

$$\phi_2(u) = \begin{cases} 0, & 0 \leqq u < r, \\ (1/\log r) - (\log u/\log r) + 1, & r \leqq u < 1, \\ 1, & u = 1, \end{cases}$$

and is a regular mass function. *In this case,* $(C, 1)$ *does not include* $[H, \alpha_1(n)]$. To see this, put $(1+z)^{-1}:\alpha_1(z) = \beta(z) = [\log (1/r)/2(1-r)] \cdot \beta_1(z-1)$, where

$$\beta_1(z) = \frac{2(1+z)}{(2+z)} \cdot \frac{1-r}{1-r^{z+1}}.$$

Inasmuch as

$$\frac{2(1+z)}{2+z} = \int_0^1 u^z d\theta_1(u),$$

$$\theta_1(u) = \begin{cases} 1, & u = 1, \\ -u^2, & 0 \leqq u < 1; \end{cases}$$

(4.1)

$$\frac{1-r}{1-r^{z+1}} = \int_0^1 u^z d\theta_2(u),$$

$$\theta_2(u) = \begin{cases} 1, & u = 1, \\ r^k, & r^k < u < r^{k-1}; \end{cases} \qquad k = 1, 2, 3, \cdots,$$

$\theta_2(u) = (1/2)[\theta_2(u+0) + \theta_2(u-0)]$ if $0 < u < 1$, we see that $\{\beta_1(n)\}$ is a regular sequence, being the product of regular sequences (cf. (ii) of §1). By means of the composition formula[10]

$$\theta(u) = \theta_1(u) + \int_0^1 \theta_2(u/v) d\theta_1(v),$$

we find that

$$\beta_1(z) = \int_0^1 u^z d\theta(u),$$

$$\theta(u) = \begin{cases} 1, & u = 1, \\ r^{k+1} - (r^{-k} - r)u^2, & r^{k+1} < u < r^k; \end{cases} \qquad k = 0, 1, 2, \cdots,$$

where $\theta(u) = (1/2)[\theta(u+0) + \theta(u-0)]$, $0 < u < 1$.

If $\{\beta(n)\}$ is a regular sequence, then we must have $\beta(n) = \int_0^1 u^n d\pi(u)$, $n = 0, 1, 2, \cdots$, and at the same time we would have:

[10] H. L. Garabedian, Einar Hille and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Mathematical Journal, vol. 8 (1941), pp. 193–213; p. 196.

$$\beta(n+1)=\frac{\log(1/r)}{2(1-r)}\int_0^1 u^n d\theta(u),\quad n=0,1,2,\cdots.$$

This implies that

$$\frac{2(1-r)}{\log(1/r)}\cdot\pi(u)=\int_0^u \frac{d\theta(t)}{t}.$$

This is impossible, for the integral on the right does not exist. In fact:

$$\int_{r^{k+1}-0}^1 \frac{d\theta(u)}{u}=2(1-r)+2r(1-r^{k+1}),$$

$$\int_{r^{k+1}+0}^1 \frac{d\theta(u)}{u}=2r(1-r^{k+1}),$$

so that $\int_t^1 d\theta(u)/u$ does not have a limit as $t\to+0$. We have proved that $\{\beta(n)\}$ is not a regular sequence, and therefore $(C,1)$ does not include $[H,\alpha_1(n)]$.

Another transformation which is suggested by Theorem 2.1 is

$$\alpha_1(z)=\frac{c+\alpha(z)}{c+1},\qquad c>0.$$

If $\alpha(z)$ is as in Theorem 2.1, then $\alpha_1(z)$ is a function of the same kind; and since the effect of the transformation is to add a discontinuity to $\phi(u)$ at $u=0$ we must have $(C,0)\approx[H,\alpha_1(n)]$.

One of the most important problems in the Hausdorff theory is the problem of recognizing whether or not a given sequence is a moment sequence. One way of contributing to this problem is to find transformations which carry moment sequences into moment sequences. *It is perhaps of interest to point out that any Hausdorff transform of a moment sequence is a moment sequence.* For, if $\{p_n\}$ is any moment sequence, and $[H,c_n]$ carries $\{p_n\}$ into $\{q_n\}$, then the relation (1.2) holds. Hence, inasmuch as $\{\Delta^n p_0\}$ is a moment sequence, and the product of two moment sequences is a moment sequence, it follows that $\{\Delta^n q_0\}$ is a moment sequence. Therefore $\{q_n\}$ is a moment sequence, as was to be proved. *The sequence* $\{q_n\}$ *is regular if and only if* $p_0c_0=1$ *and either* $c_n\to0$ *or* $\Delta^n p_0\to0$. For then and only then are the conditions (a), (b) of §1 satisfied by $\{q_n\}$. *In particular,* $\{q_n\}$ *is regular if* $\{p_n\}$, $\{c_n\}$ *are both regular.*

As an example, let $c_n=1/(n+1)$, $p_n=r^n$, $0<r<1$. Then $[H,q_n]$, where

$$q_n=\frac{1+r+r^2+\cdots+r^n}{n+1}=\frac{1-r^{n+1}}{(n+1)(1-r)},$$

is a regular Hausdorff mean. By (4.1) we see that $[H, q_n] \supset (C, 1)$. It is easy to show that $(C, 1) \supset [H, q_n]$, thereby proving that $[H, q_n] \approx (C, 1)$. If $c_n = p_n = 1/(n+1)$, then

$$q_n = \frac{1 + 2^{-1} + 3^{-1} + \cdots + n^{-1}}{n+1}$$

and $[H, q_n]$ is a regular Hausdorff mean. This mean does not include $(C, 1)$ inasmuch as $q_n : (n+1)^{-1}$ is unbounded.

NORTHWESTERN UNIVERSITY

WHIRL-SIMILITUDES, EUCLIDEAN KINEMATICS, AND NON-EUCLIDEAN GEOMETRY

J. M. FELD

1. **Introduction.** The geometry of whirls and whirl-motions in the plane had its origin in a paper by E. Kasner [6],[1] was subsequently developed in a series of papers by Kasner and DeCicco [3, 7, 8, 9], adapted to the sphere by Strubecker [10], and to 3-space by Feld [4]. In this paper we shall, by adjoining three involutory transformations, extend Kasner's whirl-motion group G_6 to a mixed group Γ_6—the *complete whirl-motion group*—composed of eight mutually exclusive, six-parameter families; these families will in turn be extended to seven-parameter families comprising the mixed group Γ_7—the *complete whirl-similitude group*. The principal results obtained are the extension of Kasner's G_6 and two representations of Γ_7: a *kinematic representation on the plane*, §6, and a *representation in quasi-elliptic 3-space*, §7.

2. **Slides, turns, and whirls.** Let the point of an oriented lineal element E have the rectangular coordinates x, y, and let the inclination of E to the x-axis be the angle θ, $0 \leqq \theta < 2\pi$. Let $z = x + iy$, $\bar{z} = x - iy$, $\zeta = e^{i\theta}$. We shall call z, ζ the *element coordinates* of E (x, y, θ), which, henceforth, shall be represented by the symbol (z, ζ).

DEFINITIONS. *A slide S_s is a lineal element transformation that translates the point of each element along its line the same distance s.*

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[1] The numbers in brackets refer to the bibliography at the end of the paper.

A turn T_α is a lineal element transformation that rotates the line of every lineal element around its point through the same angle arg α. *A direct whirl*[2] *is a product $T_\alpha S_s T_\beta$.*

From these definitions it follows that the slide S_s, $(z, \zeta) \to (z^*, \zeta^*)$, is given by the equations $z^* = z + s\zeta$, $\zeta^* = \zeta$, $(s = \bar{s})$; the turn T_α is given by the equations: $z^* = z$, $\zeta^* = \alpha\zeta$, $(|\alpha| = 1)$; and the direct whirl $T_\alpha S_s T_\beta \equiv W^+_{c,\gamma}$ is given by $z^* = z + c\zeta$, $\zeta^* = \gamma\zeta$, where $c = \alpha s$, $\gamma = \alpha\beta$, $|\gamma| = 1$.

Evidently $W^+_{a,\alpha} W^+_{b,\beta} = W^+_{c,\gamma}$ where $c = a + \alpha b$, and $\gamma = \alpha\beta$. The direct whirls form a three-parameter group $\mathfrak{W}^+_3$. The direct whirls $W^+_{c,\gamma}$ such that $c + \bar{c} = 0$ and $\gamma = \pm 1$ are dilatations.

3. **Flat fields and turbines.** A *flat field* has been defined by Kasner as a totality of ∞^2 oriented lineal elements that lie on the ∞^1 cycles (oriented circles) containing a given element called the center of the flat field. If s, σ are the coordinates of the center of the flat field F, the equation of F in current lineal element coordinates z, ζ is

$$z - s = (\bar{z} - \bar{s})\sigma\zeta. \tag{3.1}$$

Inasmuch as the parameters s, σ completely characterize F, we shall call them the coordinates of F, and use the symbol $\{s, \sigma\}$ to represent the flat field whose center is (s, σ) To distinguish the coordinates of a flat field from those of a lineal element, the former shall be referred to as *dual coordinates* and the latter as *element coordinates*. A flat field $\{s, \sigma\}$ and a lineal element (z, ζ) shall be said to be *incident* if their respective coordinates satisfy (3.1). Evidently flat fields and lineal elements can be regarded as duals of one another, and (3.1), accordingly, can be interpreted both as the equation of $\{s, \sigma\}$ in element coordinates and as the equation of (z, ζ) in dual coordinates.

Direct whirls convert flat fields into flat fields. Let $W^+_{a,\alpha}$ convert $\{s, \sigma\}$ into $\{s^*, \sigma^*\}$; then we obtain for the equations of $W^+_{a,\alpha}$ expressed in dual coordinates:

$$s^* = s - \bar{a}\sigma, \qquad \sigma^* = \bar{\alpha}\sigma. \tag{3.2}$$

Two flat fields $\{a, \alpha\}$ and $\{b, \beta\}$, $\alpha \neq \beta$, have in common ∞^1 lineal elements (z, ζ) whose coordinates satisfy the equation

$$z - l = r\zeta, \tag{3.3}$$

where $l = (a\beta - b\alpha)/(\beta - \alpha)$, $r = -(\bar{a} - \bar{b})/(\bar{\alpha} - \bar{\beta})$. The points of the

[2] The term *direct whirl* is used in place of Kasner's term *whirl* to distinguish between his and another kind of whirl which will be introduced below.

lineal elements (z, ζ) on the locus (3.3) lie on the circle $(z-l)(\bar{z}-\bar{l})=r\bar{r}$ —the *outer circle* of the locus—and their oriented lines envelope a concentric *inner cycle*. Such a locus of lineal elements has been named a *turbine* by Kasner [6]. Evidently the turbine

$$z - l = -\bar{r}\zeta \tag{3.4}$$

has the same outer circle and inner cycle as (3.3). Turbines (3.3) and (3.4) shall be called *conjugate*. A necessary and sufficient condition that the turbine (3.3) be self-conjugate is that $r+\bar{r}=0$; self-conjugate turbines, and only such turbines, are cycles.

By duality, two lineal elements (a, α) and (b, β), $\alpha \neq \beta$, have in common ∞^1 flat fields $\{s, \sigma\}$ whose coordinates satisfy the equation

$$s - \frac{a\beta - b\alpha}{\beta - \alpha} = -\frac{\bar{a} - \bar{b}}{\bar{\alpha} - \bar{\beta}}\sigma. \tag{3.5}$$

This equation represents the turbine determined by (a, α) and (b, β), $\alpha \neq \beta$, as a locus of flat fields.

Since there exists a (1, 1) correspondence between turbines and the ordered pairs of complex parameters l, r, we shall let the symbol $[l, r]$ represent the turbine whose equation in element coordinates is (3.3); l and r shall be called the *turbine coordinates* of $[l, r]$. The equation of $[l, r]$ in the dual coordinates s, σ is

$$s - l = -\bar{r}\sigma. \tag{3.6}$$

The turbine $[l, r]$ and the lineal element (z, ζ) shall be said to be *incident* if their coordinates satisfy the equation (3.3); likewise $[l, r]$ and $\{s, \sigma\}$ shall be said to be incident if their coordinates satisfy (3.6).

4. The complete group of whirls $\mathfrak{W}_3$. Let $\mathfrak{I}_1$ represent the involutory lineal element transformation $(z, \zeta) \rightarrow (z^*, \zeta^*)$, given by the equations

$$z^* = \bar{z}, \qquad \zeta^* = \bar{\zeta}. \tag{4.1}$$

The transformations $W^-_{a,\alpha} \equiv \mathfrak{I}_1 W^+_{a,\alpha}$ shall be called *opposite whirls*. Evidently $W^+_{a,\alpha}\mathfrak{I}_1 = \mathfrak{I}_1 W^+_{a,\bar{\alpha}}$. Opposite whirls convert flat fields into flat fields and turbines into turbines. In dual coordinates, s, σ, transformation $\mathfrak{I}_1$ assumes the form

$$s^* = \bar{s}, \qquad \sigma^* = \bar{\sigma} \tag{4.2}$$

and in turbine coordinates, l, r, the form

$$l^* = \bar{l}, \qquad r^* = \bar{r}. \tag{4.3}$$

The direct and opposite whirls constitute a mixed three-parameter group of which $\mathfrak{W}_3^+$ is an invariant subgroup.

Let $\mathfrak{J}_2$ designate the transformation $(z, \zeta) \rightarrow \{s^*, \sigma^*\}$ given by the equations

$$s^* = z, \qquad \sigma^* = \zeta. \tag{4.4}$$

It is easy to ascertain that the ∞^2 lineal elements (z, ζ) incident to a given flat field $\{s, \sigma\}$ are converted by $\mathfrak{J}_2$ into the ∞^2 flat fields $\{s^*, \sigma^*\}$ incident to a unique lineal element (z^*, ζ^*); the latter shall, accordingly, be regarded as the image of $\{s, \sigma\}$ produced by $\mathfrak{J}_2$. We find that $z^* = s$, $\zeta^* = \sigma$; $\mathfrak{J}_2$ is consequently an involutory transformation. The ∞^3 transformations $C_{a,\alpha}^+ \equiv \mathfrak{J}_2 W_{a,\alpha}^+$ shall be called *direct correlations*, and the ∞^3 transformations $C_{a,\alpha}^- \equiv \mathfrak{J}_2 W_{a,\alpha}^-$ shall be called *opposite correlations*. The direct and the opposite correlations transform turbines into turbines. In turbine coordinates l, r the involution $\mathfrak{J}_2$ assumes the form

$$l^* = l, \qquad r^* = -\bar{r}. \tag{4.5}$$

Consequently, $\mathfrak{J}_2$ *converts turbines into their conjugates.*

The four continuous, mutually exclusive, three-parameter families: $\mathfrak{W}_3^+$, $\mathfrak{W}_3^-$ (opposite whirls), $\mathfrak{C}_3^+$ (direct correlations), and $\mathfrak{C}_3^-$ (opposite correlations) constitute a mixed group—namely, the *complete group of whirls* $\mathfrak{W}_3$.

5. **Whirl-motions and whirl-similitudes.** Let the euclidean point displacement

$$z^* = \alpha z + a, \qquad |\alpha| = 1,$$

be extended to represent a displacement of lineal elements, as follows

$$z^* = \alpha z + a, \qquad \zeta^* = \alpha\zeta, \qquad |\alpha| = 1. \tag{5.1}$$

Let $D_{a,\alpha}^+$ denote the displacement of lineal elements given by (5.1), and let $\mathfrak{D}_3^+$ denote the group of such displacements. Furthermore, let $D_{a,\alpha}^- \equiv \mathfrak{J}_1 D_{a,\alpha}^+$ and $\mathfrak{D}_3^- \equiv \mathfrak{J}_1 \mathfrak{D}_3^+$; $\mathfrak{D}_3^-$ evidently denotes the family of euclidean symmetries operating on lineal elements. The group of euclidean motions $\mathfrak{D}_3$ $(= \mathfrak{D}_3^+ + \mathfrak{D}_3^-)$ converts flat fields into flat fields and turbines into turbines. The product of a direct whirl by a displacement is commutative. Let such a product be called a *direct whirl-motion.* The direct whirl-motions constitute a continuous, six-parameter group which shall be denoted by $\mathfrak{G}_6^1$.

Let

$$\mathfrak{G}_6^2 \equiv \mathfrak{W}_3^+ \mathfrak{D}_3^-, \qquad \mathfrak{G}_6^3 \equiv \mathfrak{C}_3^+ \mathfrak{D}_3^+, \qquad \mathfrak{G}_6^4 \equiv \mathfrak{C}_3^- \mathfrak{D}_3^+.$$

Evidently $\mathfrak{G}_6^2=\mathfrak{J}_1\mathfrak{G}_6^1$; $\mathfrak{G}_6^3=\mathfrak{J}_2\mathfrak{G}_6^1$; $\mathfrak{G}_6^4=\mathfrak{J}_1\mathfrak{G}_6^3$. The four families $\mathfrak{G}_6^i$ are mutually exclusive and constitute a mixed six-parameter group $\mathfrak{G}_6$, which shall be called the *group of proper whirl-motions*. The two families $\mathfrak{G}_6^1$, $\mathfrak{G}_6^2$ form a mixed invariant subgroup of $\mathfrak{G}_6$.

Let $\mathfrak{J}_3$ denote the involutory lineal element transformation

$$z^* = z\bar{\zeta}, \qquad \zeta^* = \bar{\zeta}. \tag{5.2}$$

Evidently $\mathfrak{J}_3$ transforms flat fields into flat fields and turbines into turbines. In dual coordinates $\mathfrak{J}_3$ is given by the equations

$$s^* = -\bar{s}\sigma, \qquad \sigma^* = \sigma, \tag{5.3}$$

and in turbine coordinates l, r, by the equations

$$l^* = r, \qquad r^* = l. \tag{5.4}$$

Let $\mathfrak{H}_6^i \equiv \mathfrak{J}_3\mathfrak{G}_6^i$, $i=1, 2, 3, 4$. The four, mutually exclusive, continuous families of turbine-preserving transformations $\mathfrak{H}_6^i$ comprise the *improper whirl-motions*. The totality of proper and improper whirl-motions constitute a mixed group composed of eight continuous families, which shall be called the *complete group of whirl-motions*, and denoted by Γ_6.

Let $\mathfrak{M}$ denote the magnification $(z, \zeta)\rightarrow(z^*, \zeta^*)$ given by

$$z^* = kz, \qquad \zeta^* = \zeta, \tag{5.5}$$

where k is real and positive. Let $\Gamma_7 \equiv \mathfrak{M}\Gamma_6$. Evidently Γ_7 is a mixed seven-parameter group of turbine-preserving transformations comprising the eight continuous families: $\mathfrak{G}_7^i \equiv \mathfrak{M}\,\mathfrak{G}_6^i$, $\mathfrak{H}_7^i \equiv \mathfrak{M}\,\mathfrak{H}_6^i$, $i=1, 2, 3, 4$; Γ_7 shall be called the *complete group of whirl-similitudes*.

6. **The kinematic representation.** Owing to the (1, 1) correspondence subsisting between turbines $\mathfrak{T}$: $[l, r]$ and ordered pairs of complex numbers l, r, $\mathfrak{T}$ can be mapped upon the ordered pair of points on the Gauss plane represented by l and r in this order. This mapping, $\mathfrak{T}\rightarrow(l, r)$, of the ∞^4 turbines upon the ∞^4 ordered point pairs shall be called a *kinematic representation*. With the aid of equations (3.3) and (3.6), which entail necessary and sufficient conditions for the incidence of turbine and lineal element in the former case, and for the incidence of turbine and flat field in the latter, the following two theorems can now be established.

THEOREM 1. *The kinematic* (1, 1) *representation of turbines by ordered pairs of real points on a plane, namely* $\mathfrak{T}\leftrightarrow(l, r)$, *induces a* (1, 1) *correspondence between lineal elements* (z, ζ) *and planar euclidean displace-*

ments $l \to r$. To the elements (z, ζ) such that $\zeta = -1$ correspond the translations $r = l - z$, and to those such that $\zeta \neq -1$ correspond the rotations around the point $z/(\zeta+1)$ through the angle arg $(-\bar{\zeta})$.

THEOREM 2. *The kinematic (1, 1) representation $\mathfrak{T} \leftrightarrow (l, r)$ induces a (1, 1) correspondence between flat fields $\{s, \sigma\}$ and planar euclidean symmetries $l \to r$. To the flat fields $\{s, \sigma\}$ such that $\sigma\bar{s} + s = 0$ correspond reflections $l \to r$ in the line $\bar{s}z + s\bar{z} = s\bar{s}$; and to those such that $\sigma\bar{s} + s \neq 0$ correspond symmetries compounded, in either order, of a reflection in the axis $z - s/2 = \sigma(\bar{z} - \bar{s}/2)$ and a translation (parallel to the axis) given by $r = l - (\sigma\bar{s} + s)/2$.*

The proper whirl-similitudes—that is, those constituting the four families $\mathfrak{G}_7^i$—regarded as turbine transformations $\mathfrak{T} \to \mathfrak{T}^*$, are mapped by means of the kinematic representation upon pairs of similitudes $l \to l^*$, $r \to r^*$ having the same ratio of magnification. These pairs of similitudes form four continuous families corresponding to the $\mathfrak{G}_7^i$; their equations are as follows:

$$\mathfrak{G}_7^1: \quad l^* = \alpha k l + a, \qquad r^* = \beta k r + b,$$

$$\mathfrak{G}_7^2: \quad l^* = \alpha k \bar{l} + a, \qquad r^* = \beta k \bar{r} + b,$$

$$\mathfrak{G}_7^3: \quad l^* = \alpha k l + a, \qquad r^* = \beta k \bar{r} + b,$$

$$\mathfrak{G}_7^4: \quad l^* = \alpha k \bar{l} + a, \qquad r^* = \beta k r + b,$$

$$k \text{ real and positive}, \quad |\alpha| = |\beta| = 1.$$

By interchanging l and r in the right-hand members of the equations of the similitudes corresponding to $\mathfrak{G}_7^i$, we obtain the equations of the kinematic image of the family of improper whirl-similitudes $\mathfrak{H}_7^i$. The kinematic images of the eight families comprising Γ_8 are the pairs of euclidean motions to which the above pairs of similitudes reduce when $k = 1$.

7. **The quasi-elliptic representation.** Let a point $\mathfrak{p}$ (X, Y, Z) in S_3 have the homogeneous coordinates $X = p_2/p_1$, $Y = p_3/p_1$, $Z = p_0/p_1$. Let a line C, together with the pair of conjugate imaginary points on C, namely:

$$c_l, \qquad p_0 : p_1 : p_2 : p_3 = 0 : 0 : i : 1,$$

$$c_r, \qquad p_0 : p_1 : p_2 : p_3 = 0 : 0 : 1 : i,$$

and the pair of conjugate imaginary planes through C:

$$\gamma_l, \quad p_0 - ip_1 = 0, \qquad \gamma_r, \quad p_0 + ip_1 = 0$$

be the Cayley absolute underlying a non-euclidean space Q_3, called

quasi-elliptic by Blaschke [1, 2]. The proper (real) points of Q_3 are those that do not lie on C; the proper lines are those that do not intersect C; and the proper planes are those that do not pass through C.

By means of a method independently discovered by J. Grünwald [5] and W. Blaschke [1, 2], the ∞^4 proper (real) lines $\mathfrak{g}$ of Q_3 can be mapped continuously and (1, 1) upon the ∞^4 pairs of ordered real points (l, r) of the euclidean ground plane $Z=0$ in a manner such that the following two properties subsist: (1) the two fields of points l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ passing through a proper real point $\mathfrak{p}$ are such that a unique euclidean displacement $l \rightarrow r$ will transform the points l into their associated points r; (2) the two fields l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ lying in a proper real plane π are such that a unique euclidean symmetry $l \rightarrow r$ will transform the points l into their associated points r. By virtue of these two properties a (1, 1) correspondence exists between the points $\mathfrak{p}$ of Q_3 and the euclidean displacements in the ground plane on the one hand, and between the planes π and the euclidean symmetries in the ground plane on the other hand.

Let us associate with every proper real line $\mathfrak{g}$ in Q_3 a turbine $\mathfrak{T}$ in such a manner that $\mathfrak{g}$ is the *B-G* (Blaschke-Grünwald) image of that pair of points l, r as that of which $\mathfrak{T}$ is the kinematic image. The ensuing correspondence $\mathfrak{g} \leftrightarrow \mathfrak{T}$ is (1, 1) and continuous. Furthermore, let us establish a (1, 1) correspondence between lineal elements $\mathcal{E}$ and proper points $\mathfrak{p}$ in Q_3 such that $\mathcal{E}$ and $\mathfrak{p}$ are images—kinematic and *B-G*, respectively,—of the same euclidean displacement $l \rightarrow r$; and, finally, let us establish a third (1, 1) correspondence between flat fields $\mathcal{F}$ and proper planes π such that $\mathcal{F}$ and π are images of the same symmetry $l \rightarrow r$. These three (1, 1) correspondences define the *quasi-elliptic representation* of whirl-similitude geometry. Incidences among lineal elements, turbines, and flat fields imply corresponding incidences among their quasi-elliptic images.

Inasmuch as the group $\overline{\mathfrak{G}}_7^1$ of collineations in Q_3 that leave the points c_l, c_r and the planes γ_l, γ_r of the absolute individually invariant corresponds, by virtue of the *B-G* mapping, to the group of transformations $(l, r) \rightarrow (l^*, r^*)$ such that $l \rightarrow l^*$, $r \rightarrow r^*$ are two similitudes having the same ratio of magnification [1, 2], we see that, by virtue of the results given in §6, $\overline{\mathfrak{G}}_7^1$ is simply isomorphic to $\mathfrak{G}_7^1$. The totality of automorphisms (collineations and correlations) of Q_3 constitutes a mixed group $\overline{\Gamma}_7$, composed of eight continuous families, $\overline{\mathfrak{G}}_7^i$ and $\overline{\mathfrak{H}}_7^i$, $i=1, 2, 3, 4$, which are isomorphic to the families comprising the complete group of whirl-similitudes Γ_7. To characterize the isomorphism

$\Gamma_7 \leftrightarrow \overline{\Gamma}_7$, it is sufficient to indicate the quasi-elliptic representations of $\mathfrak{I}_1$, $\mathfrak{I}_2$, and $\mathfrak{I}_3$. These are, in terms of the involutory transformations effected on the absolute, as follows

$$\mathfrak{I}_1\colon\ c_l \to c_r,\quad c_r \to c_l,\quad \gamma_l \to \gamma_r,\quad \gamma_r \to \gamma_l;$$
$$\mathfrak{I}_2\colon\ c_l \to \gamma_l,\quad c_r \to \gamma_r,\quad \gamma_l \to c_l,\quad \gamma_r \to c_r;$$
$$\mathfrak{I}_3\colon\ c_l \to c_l,\quad c_r \to c_r,\quad \gamma_l \to \gamma_r,\quad \gamma_r \to \gamma_l.$$

To $\mathfrak{G}_6^1$ (Kasner's whirl-motion group) there corresponds in quasi-elliptic space, by virtue of the quasi-elliptic representation, the group of *quasi-motions* $\overline{\mathfrak{G}}_6^1$. The quasi-motions can be uniquely resolved into products of what Blaschke calls, because of their analogy to Clifford's left and right translations in elliptic space, *quasi-left and quasi-right* translations in Q_3 [1]; the former are the quasi-elliptic images of the displacements $\mathfrak{D}_3^+$, and the latter are the images of the direct whirls $\mathfrak{W}_3^+$.

By means of the quasi-elliptic representation, many of the results obtained by Kasner and DeCicco for the geometry of whirl-motions can be easily identified with results obtained independently by Blaschke and Grünwald for quasi-elliptic space.

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Queens College and
Brooklyn College